Foundations of Physics, Vol. 8, Nos. 1/2, 1978

Generalized Two-Level Quantum Dynamics. II. Non-Hamiltonian State Evolution

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Received July 8, 1976

A theorem is derived that enables a systematic enumeration of all the linear superoperators $\mathscr{L}$ (associated with a two-level quantum system) that generate, via the law of motion $\mathscr{L}\rho = \dot{\rho}$, mappings $\rho(0) \to \rho(t)$ restricted to the domain of statistical operators. Such dynamical evolutions include the usual Hamiltonian motion as a special case, but they also encompass more general motions, which are noncyclic and feature a destination state $\rho(t \to \infty)$ that is in some cases independent of $\rho(0)$.

1. INTRODUCTION

We discussed in a previous publication[1] (part I of this series) the physical rationale for considering the theoretical possibility of generalizing the usual dynamical postulate of quantum mechanics so that it would describe, even for an isolated system, both reversible and irreversible motions. The idea is to retain the basic Liouvillian form

$$\mathscr{L}\rho = \dot{\rho} \tag{1}$$

where ρ is the usual statistical operator, but $\mathscr{L}$, the Liouville superoperator, is *not* to be constrained to the (reversible) Hamiltonian form

$$\mathscr{L}^H\rho = (1/i)[H, \rho] \tag{2}$$

It is essential, however, to demand that an $\mathscr{L}$, to be admissible, must belong to the class of linear superoperators that, through (1), invariably generate mappings $\rho(0) \to \rho(t)$ called *dynamical evolutions*, such that if $\rho(0)$ is a statistical operator, then $\rho(t)$ is also.

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For the case of a two-dimensional Hilbert space $\mathscr{H}_2$, we found necessary and sufficient characterizations of admissible $\mathscr{L}$'s in two different representations. Specifically, if a dyadic "quorum" (a term we use to denote a basis for the space $\mathscr{A}$ of operators on the Hilbert space) $\{Q_{nm}\}$ is adopted, i.e.,

$$\{Q_{nm}\} \equiv \{| \alpha_n\rangle\langle\alpha_m | \mid m, n = 1, 2\} \tag{3}$$

where $\{| \alpha_n\rangle\}$ is a complete orthonormal basis for $\mathscr{H}_2$, then the matrix elements of $\mathscr{L}$, defined by

$$\mathscr{L}_{mn,kl} \equiv \mathrm{Tr}(Q_{mn}^{\dagger}\mathscr{L}Q_{kl}) \tag{4}$$

must satisfy the following conditions:

$$\begin{aligned}
\mathscr{L}'_{ff,ff} &= \sum_{mnkl} U_{ff,ff;mn,kl}(a, b)\, \mathscr{L}_{mn,kl} \leqslant 0, && f = 1, 2 \\
\mathscr{L}'_{ff,gg} &= \sum_{mnkl} U_{ff,gg;mn,kl}(a, b)\, \mathscr{L}_{mn,kl} \geqslant 0, && f \neq g \\
\sum_{f=1}^{2} \mathscr{L}'_{ff,gg} &= \sum_{f=1}^{2} \sum_{mnkl} U_{ff,gg;mn,kl}(a, b)\, \mathscr{L}_{mn,kl} = 0, && g = 1, 2
\end{aligned} \tag{5}$$

for every a, b such that $| a |^2 + | b |^2 = 1$, where

$$U_{fg,ij;mn,kl}(a, b) \equiv U_{fm}^{*} U_{gn} U_{ik} U_{jl}^{*} \tag{6}$$

with

$$(U_{mn}) \equiv \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \tag{7}$$

On the other hand, if the orthonormal Hermitian quorum

$$\{\nu_\alpha \mid \alpha = 0, 1, 2, 3\} \equiv \left(\frac{1}{\sqrt{2}}, \frac{\boldsymbol{\sigma}}{\sqrt{2}}\right) \tag{8}$$

where

$$\mathrm{Tr}(\nu_\alpha \nu_\beta) = \delta_{\alpha\beta} \tag{9}$$

is chosen, $\boldsymbol{\sigma}$ being the Pauli matrices, then the necessary and sufficient conditions on $\mathscr{L}$ take a simpler form. The matrix elements of $\mathscr{L}$, defined by

$$\mathscr{L}_{\beta\alpha} = \mathrm{Tr}(\nu_\beta \mathscr{L} \nu_\alpha) \tag{10}$$

must be real and must also satisfy these conditions:

$$\mathscr{L}_{0\alpha} = 0 \tag{11}$$

and

$$\mathrm{Tr}(P\mathscr{L}P) \leqslant 0 \tag{12}$$

for every one-dimensional projector P.

In order to extract from these mathematical conditions a more physical classification of the types of motion that the generalized dynamical postulate (1) is capable of describing, we develop next a geometrical interpretation of (11) and (12).

2. LIOUVILLIANS AND ASSOCIATED QUADRIC SURFACES

Consider the quantity

$$G \equiv 2\,\mathrm{Tr}(X\mathscr{L}X) \tag{13}$$

where X is a Hermitian operator of the form

$$X = (1/\sqrt{2})(\nu_0 + \mathbf{s}\cdot\boldsymbol{\nu}) \tag{14}$$

We may conveniently regard $\mathbf{s}$ as an element of an auxiliary real 3-space $\mathscr{S}$. Note that if $\mathbf{s}\cdot\mathbf{s} \leqslant 1$, then X is a statistical operator (Hermitian, trace unity, nonnegative definite); if $\mathbf{s}$ is on the unit sphere in $\mathscr{S}$, X is a one-dimensional projector. These correspondences between statistical operators, including projectors, and points of $\mathscr{S}$ are one-to-one.

Now, using (9)–(11), and (14) in (13), we obtain

$$G = \sum_{m=1}^{3}\sum_{n=1}^{3} \mathscr{L}_{mn} s_m s_n + \sum_{n=1}^{3} \mathscr{L}_{n0} s_n \tag{15}$$

The quadratic form (15) represents, for each value of G, a quadric surface in $\mathscr{S}$ whose exact shape depends on the values of $\mathscr{L}_{mn}$, $\mathscr{L}_{n0}$, and G. We shall call the case $G = 0$ the *critical* surface; note that it passes through the origin of $\mathscr{S}$.

The necessary and sufficient conditions (11) and (12) for $\mathscr{L}$ to generate a dynamical evolution are obviously equivalent to (11) plus the algebraic statement that $G \leqslant 0$ for every $\mathbf{s}$ satisfying $\mathbf{s}\cdot\mathbf{s} = 1$. This analytic characterization of admissible $\mathscr{L}$'s for $\mathscr{H}_2$ has been given previously by Kossakowski.[2] In the present investigation we employ instead a geometrical interpretation of (11) and (12): $\mathscr{L}_{\beta\alpha}$ are the matrix elements of an admissible $\mathscr{L}$ if and only if $\mathscr{L}_{0\alpha} = 0$ and *each quadric surface* (15) *for which* $G > 0$ *does not intersect the unit sphere in* $\mathscr{S}$.

To apply this criterion, we examine (15) using standard methods of analytic geometry. Define K_{mn} as the symmetric part of $\mathscr{L}_{mn}$, J_{mn} as the antisymmetric part; then

$$K_{mn} = \tfrac{1}{2}(\mathscr{L}_{mn} + \mathscr{L}_{nm}), \qquad J_{mn} = \tfrac{1}{2}(\mathscr{L}_{mn} - \mathscr{L}_{nm}) \tag{16}$$

Also let

$$B_n = -\mathscr{L}_{n0} \tag{17}$$

When (16) and (17) are substituted into (15), the antisymmetric part drops out, leaving

$$G = \sum_{mn} K_{mn} s_m s_n - \sum_n B_n s_n \tag{18}$$

Let R be the rotation in $\mathscr{S}$ space that diagonalizes K_{mn}; then the new coordinates $\{x_m\}$ are related to $\{s_n\}$ by

$$s_n = \sum_m R_{nm} x_m \tag{19}$$

the new components of vector $\{B_m\}$ are

$$C_n \equiv \sum_m R_{mn} B_m \tag{20}$$

and the eigenvalues of K_{mn} are

$$K_p \equiv \sum_{mn} R_{mp} R_{np} K_{mn} \tag{21}$$

The quadratic form (15) thus becomes

$$G = \sum_n K_n x_n^2 - \sum_n C_n x_n \tag{22}$$

Using (22), we can easily demonstrate the necessity of the condition

$$K_n \leqslant 0 \tag{23}$$

if $G \leqslant 0$ is demanded whenever $\mathbf{x}$ is on the unit sphere. Thus, if $x_n = \delta_{nj}$, we obtain

$$G = K_j - C_j \leqslant 0 \tag{24}$$

while $x_n = -\delta_{nj}$ gives

$$G = K_j + C_j \leqslant 0 \tag{25}$$

Adding the inequalities (24) and (25) immediately yields the result (23), which restricts considerably the shapes of quadric surfaces that may be associated with admissible Liouvillian superoperators. For example, with (23) in force, (22) cannot describe hyperboloids. Stated another way, the restriction (23) implies that the symmetric part of $\mathscr{L}_{mn}$ must be a negative-semidefinite matrix.

Next we seek to enumerate systematically all values of $\{K_n\}$ and $\{C_n\}$ for which the corresponding $\mathscr{L}$ generates a dynamical evolution.

3. THE ELLIPSOID CASE

In this section we assume that all three K_n are strictly negative. It is then permissible to rearrange (22) to obtain

$$\sum_n K_n \left(x_n - \frac{C_n}{2K_n}\right)^2 = G + \frac{1}{4}\sum_n \frac{C_n^2}{K_n} \tag{26}$$

There are obviously no points $\mathbf{x}$ for which the lhs of (26) is positive; hence we are concerned only with

$$G + \frac{1}{4}\sum_n \frac{C_n^2}{K_n} \leqslant 0 \tag{27}$$

When the equality holds in (27), then G is positive, since $K_n < 0$, and (26) represents only the single point $\mathscr{E}_0$ with coordinates $\{C_n/2K_n\}$. We must therefore demand that this point not lie on the unit sphere.

When the strict inequality holds in (27), we may rewrite (26) as

$$\sum_n \frac{[x_n - (C_n/2K_n)]^2}{(1/K_n)[G + \frac{1}{4}\sum_m (C_m^2/K_m)]} = 1 \tag{28}$$

which is a family of ellipsoids parameterized by G, all centered at $\mathscr{E}_0$. The limiting case $G \to -\frac{1}{4}\sum_n (C_n^2/K_n)$ gives the single point discussed above. As G decreases from this maximum value, each successive ellipsoid contains all preceding ones. Thus all the $G > 0$ ellipsoids are inside the critical ($G = 0$) ellipsoid, and all $G < 0$ ellipsoids are outside the critical ellipsoid. It follows that the geometrical criterion for admissible $\mathscr{L}$'s, viz., that each $G > 0$ quadric surface not intersect the unit sphere, will be satisfied in this case if and only if the critical ellipsoid has no points external to the unit sphere.

To reformulate this geometrical constraint analytically, let

$$g(\mathbf{x}) \equiv \sum_n K_n x_n^2 - \sum_n C_n x_n \tag{29}$$

Then

$$g(\mathbf{x}) = 0 \tag{30}$$

is the equation of the critical ellipsoid and $\nabla g(\mathbf{x})$ is a vector normal to the ellipsoid at a surface point $\mathbf{x}$. Since the critical ellipsoid passes through the origin, the points on the critical ellipsoid that are farthest from the origin will satisfy the equation

$$\mathbf{x} \times \nabla g(\mathbf{x}) = 0 \tag{31}$$

i.e., the radius vectors locating such points will be parallel (or antiparallel) to the surface normals at those points. We may therefore express the criterion that the critical ellipsoid shall have no points external to the unit sphere by requiring

$$\mathbf{x}_0 \cdot \mathbf{x}_0 \leqslant 1 \tag{32}$$

for all $\mathbf{x}_0$ satisfying both (30) and (31). [There will of course sometimes be points $\mathbf{x}_0$ other than those farthest from the origin that satisfy (30) and (31), but this in no way affects the validity of the condition (32), which must in any case hold for all such $\mathbf{x}_0$.]

Equation (31) is equivalent to the three relations

$$\begin{aligned} 2yz(K_3 - K_2) + C_2 z - C_3 y &= 0 \\ 2xz(K_3 - K_1) + C_1 z - C_3 x &= 0 \\ 2xy(K_2 - K_1) + C_1 y - C_2 x &= 0 \end{aligned} \tag{33}$$

where

$$(x, y, z) \equiv (x_1, x_2, x_3) \tag{34}$$

If the first equation in (33) is multiplied by x, the second by y, and the two are then subtracted, the result is the third equation multiplied by z. Thus (33) is a functionally dependent set, only two of which are actually independent conditions.

We conclude that if $K_1, K_2, K_3 < 0$, then the C_n must be chosen so that every solution $\mathbf{x}_0$ of (30) and (33) satisfies (32). It is geometrically obvious that there are in general many acceptable critical ellipsoids.

4. THE ELLIPTIC-PARABOLOID AND ELLIPTIC-CYLINDER CASES

The next case to be considered is when one eigenvalue of K_{mn} vanishes, the other two being strictly negative. To be explicit, we take $K_1, K_2 < 0$, $K_3 = 0$. The quadratic form (22) of interest then becomes

$$G = \sum_n^2 K_n x_n^2 - \sum_n C_n x_n^2 - C_3 x_3 \tag{35}$$

which may be rearranged as

$$\sum_{n}^{2} K_n \left(x_n - \frac{C_n}{2K_n}\right)^2 = C_3 x_3 + \left(G + \frac{1}{4}\sum_{n}^{2} \frac{C_n^2}{K_n}\right) \tag{36}$$

If $C_3 \neq 0$, we can obtain from (36) the form

$$\sum_{n}^{2} \frac{[x_n - (C_n/2K_n)]^2}{-1/K_n} = \frac{x_3 - \{-(1/C_3)[G + \frac{1}{4}\sum_n^2 (C_n^2/K_n)]\}}{-1/C_3} \tag{37}$$

which describes a family of elliptic paraboloids parametrized by G. The vertices of these paraboloids all lie on a straight line $\mathscr{E}_1$ parallel to the x_3 axis and intersecting the $x_1 x_2$ plane at $(C_1/2K_1, C_2/2K_2)$. The $G = 0$ paraboloid, which passes through the origin, divides the 3-space into two regions, one containing all $G > 0$, the other all $G < 0$ paraboloids. It is therefore impossible to avoid intersections of $G > 0$ paraboloids with the unit sphere, and we must conclude that no acceptable Liouvillians $\mathscr{L}$ are associated with $K_1, K_2 < 0$, $K_3 = 0$, $C_3 \neq 0$.

If $C_3 = 0$, we reduce (36) to

$$\sum_{n}^{2} K_n \left(x_n - \frac{C_n}{2K_n}\right)^2 = G + \frac{1}{4}\sum_{n}^{2} \frac{C_n^2}{K_n} \tag{38}$$

The lhs of (38) cannot be positive, so we are interested only in values of G such that

$$G + \frac{1}{4}\sum_{n}^{2} \frac{C_n^2}{K_n} \leqslant 0 \tag{39}$$

When the equality holds in (39), the expression (38) represents the straight line $\mathscr{E}_1$ described above. If the strict inequality holds in (39), we may rearrange (38) to obtain

$$\sum_{n}^{2} \frac{[x_n - (C_n/2K_n)]^2}{(1/K_n)[G + \frac{1}{4}\sum_m^2 (C_m^2/K_m)]} = 1 \tag{40}$$

which describes a family of elliptic cylinders all centered on the line $\mathscr{E}_1$. As usual, the critical cylinder $G = 0$ passes through the origin; all $G > 0$ cylinders are contained within the critical cylinder, while all $G < 0$ cylinders are outside. Thus if the critical cylinder has an interior, it will be impossible to prevent intersections of $G > 0$ cylinders with the unit sphere. This implies that the straight line $\mathscr{E}_1$ must itself be the (degenerate) critical cylinder; i.e.,

$$G = -\frac{1}{4}\sum_{n}^{2} \frac{C_n^2}{K_n} = 0 \tag{41}$$

which means that $C_1 = C_2 = 0$.

We conclude that the case $K_1, K_2 < 0$, $K_3 = 0$ is of physical interest only if $C_1 = C_2 = C_3 = 0$. By symmetry we infer that if one and only one K_n vanishes, all C_n must vanish.

5. THE PARABOLIC CYLINDER AND PLANE CASES

Let only one eignenvalue of K_{mn} be nonvanishing; e.g., $K_1 = K_2 = 0$, $K_3 < 0$. The basic quadratic form is then

$$G = K_3 x_3^2 - C_3 x_3 - \sum_n^2 C_n x_n \tag{42}$$

or

$$K_3\left(x_3 - \frac{C_3}{2K_3}\right)^2 = G + \sum_n^2 C_n x_n + \frac{C_3^2}{4K_3} \tag{43}$$

If we now rotate axes about the x_3 axis until the vector $\mathbf{C}$ with old components (C_1, C_2, C_3) has new components (D, O, C_3), the transform of (43) is

$$K_3\left(y_3 - \frac{C_3}{2K_3}\right)^2 = Dy_1 + G + \frac{1}{4}\frac{C_3^2}{K_3} \tag{44}$$

where $\{y_n\}$ are the new coordinates.

If $D \neq 0$, we can rearrange (44) to obtain

$$\left(y_3 - \frac{C_3}{2K_3}\right)^2 = \frac{D}{K_3}\left[y_1 - \left(-\frac{G + \frac{1}{4}(C_3^2/K_3)}{D}\right)\right] \tag{45}$$

which describes a family of parabolic cylinders. Again the critical cylinder $G = 0$ divides the space into $G < 0$ and $G > 0$ regions, and it is impossible to keep $G \leqslant 0$ on the unit sphere.

If $D = 0$, then $C_1 = C_2 = 0$ and (43) becomes

$$K_3\left(x_3 - \frac{C_3}{2K_3}\right)^2 = G + \frac{C_3^2}{4K_3} \tag{46}$$

Since the lhs cannot be positive, we have

$$G + \frac{C_3^2}{4K_3} \leqslant 0 \tag{47}$$

and (46) may be solved to obtain

$$x_3 = \frac{C_3}{2K_3} - \left[\frac{1}{K_3}\left(G + \frac{C_3^2}{4K_3}\right)\right]^{1/2} \tag{48}$$

which describes a family of planes perpendicular to the x_3 axis. The critical plane $G = 0$ divides the space into $G < 0$ and $G > 0$ regions. It is therefore impossible to keep $G \leqslant 0$ on the unit sphere unless $C_3 = 0$, in which case (46) degenerates to

$$K_3 x_3^2 = G \tag{49}$$

and $G \leqslant 0$ everywhere.

Accordingly, we conclude that the case $K_1 = K_2 = 0$, $K_3 < 0$ is associated with dynamical evolution only if $C_1 = C_2 = C_3 = 0$. Again by symmetry we may infer that if one and only one K_n is nonvanishing all C_n must vanish.

We have so far omitted the case where $K_1 = K_2 = K_3 = 0$. Here the quadratic form degenerates to

$$G = -\sum_n C_n x_n \tag{50}$$

There is no ellipsoid, only a family of parallel planes whose critical member $G = 0$ divides the space into $G < 0$ and $G > 0$ regions. There is again no way to maintain $G \leqslant 0$ on the unit sphere unless all the $C_n = 0$.

Having considered now all possible cases, we may summarize our findings as follows. In the ellipsoid case, where none of the K_n vanishes, it is generally possible to find nonzero C_n such that the corresponding Liouville superoperator is acceptable; in all other cases, where one or more of the K_n vanish, it is always necessary and sufficient that all $C_n = 0$. We have then the following general result.

Theorem: A superoperator $\mathscr{L}$ generates a dynamical evolution for a two-level system if and only if (a) $\mathscr{L}_{0\alpha} = 0$; (b) K_{mn}, the symmetric part of $\mathscr{L}_{mn}$, is a negative-semidefinite matrix; and (c) $\mathscr{L}_{m0}$ meets these requirements: (i) if $\det(K_{mn}) < 0$, $\mathscr{L}_{m0}$ must be associated via (17) and (20) with C_m such that (32) is satisfied; or, (ii) if $\det(K_{mn}) = 0$, $\mathscr{L}_{m0} = 0$.

6. HAMILTONIAN AND NON-HAMILTONIAN PARTS OF $\mathscr{L}$

To obtain the traditional Hamiltonian form of quantum dynamics we can interpret the equation of motion

$$(1/i)[H, \rho] = \dot{\rho} \tag{51}$$

as an example of Liouvillian evolution,

$$\mathscr{L}^H \rho = \dot{\rho} \tag{52}$$

with the superoperator generator defined by

$$\mathscr{L}^H \rho = (1/i)[H, \rho] \tag{53}$$

In terms of some *arbitrary* dyadic quorum, we write the Hamiltonian

$$H = \sum_{ij} h_{ij} \mid \alpha_i \rangle \langle \alpha_j \mid \tag{54}$$

where

$$h_{ij} = \langle \alpha_i \mid H \mid \alpha_j \rangle \tag{55}$$

and hence

$$\mathscr{L}^H_{kl,nm} = \mathrm{Tr}(Q_{kl}^\dagger \mathscr{L}^H Q_{nm}) = -i(h_{kn}\delta_{ml} - \delta_{kn}h_{ml}) \tag{56}$$

The relevant components needed to verify the conditions (5) for the case $a = 1$, $b = 0$ are

$$L_{11,11} = 0, \qquad L_{22,22} = 0, \qquad L_{11,22} = 0, \qquad L_{22,11} = 0 \tag{57}$$

It is obvious that since the representation was arbitrary, the same zero values will be found in all representations, and hence for all a, b, (5) will be satisfied by $\mathscr{L}^H$. It is a routine chore to verify this by performing the transformations described by (5)–(7). Thus, as expected, Hamiltonian evolution is a trivial illustration of the generalized dynamical evolution we wish to study.

To see where the Hamiltonian-type Liouvillian fits into the complete family of admissible supergenerators described in the theorem in the preceding section, we first find the matrix elements of $\mathscr{L}^H$ in the $\{\nu_\alpha\}$ quorum.

Using (10) and (53), we obtain

$$\mathscr{L}^H_{\beta\alpha} = \mathrm{Tr}\left(\nu_\beta \frac{1}{i}[H, \nu_\alpha]\right) = \frac{\sqrt{2}}{i} \sum_\mu h_\mu \, \mathrm{Tr}(\nu_\beta[\nu_\mu, \nu_\alpha]) \tag{58}$$

where

$$H = \sqrt{2} \sum_\mu h_\mu \nu_\mu \tag{59}$$

Recalling (8), (9), and the commutation properties of the Pauli matrices, we find from (58) that

$$\mathscr{L}^H_{0\alpha} = \mathscr{L}^H_{\beta 0} = 0 \tag{60}$$

and

$$\mathscr{L}^H_{nk} = 2 \sum_m h_m \sum_l \epsilon_{mkl} \delta_{nl} = 2 \sum_m h_m \epsilon_{mkn} \tag{61}$$

where ϵ_{mkn} is the skew-symmetric Levi-Civita symbol. In matrix form, we have therefore

$$(\mathscr{L}^H_{\beta\alpha}) = 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -h_3 & h_2 \\ 0 & h_3 & 0 & -h_1 \\ 0 & -h_2 & h_1 & 0 \end{pmatrix} \tag{62}$$

Inspection of (62) readily shows that Hamiltonian-type dynamical evolution corresponds to the simplest case considered in our general analysis—all the K_n vanish. Thus the critical quadric surface for $\mathscr{L}^H$ is merely the point at the origin of $\mathscr{S}$.

For the most general $\mathscr{L}$, it is apparent that the following decomposition always exists uniquely:

$$(\mathscr{L}_{\beta\alpha}) = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & (J_{mn}) & \\ 0 & & & \end{array}\right) + \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline & & & \\ -(B_n) & & (K_{mn}) & \\ & & & \end{array}\right) \tag{63}$$

where B_n, J_{mn}, and K_{mn} are defined as in (16) and (17). We may therefore define the first term in (63) as the *Hamiltonian part* of the given $\mathscr{L}$, the corresponding H being determined by substituting into (59) the h_r given by

$$h_r = -\tfrac{1}{4} \sum_{mn} \epsilon_{rmn} J_{mn} \tag{64}$$

which follows from (61). Note that h_0 is not determined by $\mathscr{L}$. (This occasions no concern, however, since h_0 plays no role in generating Hamiltonian motion; it only determines the value of, say, the lower energy level.)

The second term in (63), the *non-Hamiltonian part* of $\mathscr{L}$, contains the parameters whose admissible values are given by the theorem in Section 5.

It is also possible to write decompositions like (63) using the dyadic quorum $\{Q_{mn}\} \equiv \{|\,\alpha_m\rangle\langle\alpha_n\,|\}$. If the superoperator $\mathscr{L}$ is first restricted by the two obvious constraints, implied in the conditions (5), that $\mathscr{L}\rho = \dot{\rho}$ has to be Hermitian and to have a zero trace, we find after a rather tedious argument that in this sense the most general form of $\mathscr{L}$ is

$$(\mathscr{L}_{mn,kl}) = \frac{1}{i}\begin{pmatrix} 0 & \theta^* & -\theta & 0 \\ \theta & \Delta & 0 & -\theta \\ -\theta^* & 0 & -\Delta & \theta^* \\ 0 & -\theta^* & \theta & 0 \end{pmatrix} + \frac{1}{i}\begin{pmatrix} -ip & \eta & -\eta^* & iq \\ \epsilon & \phi & -\lambda^* & \tau \\ -\epsilon^* & \lambda & -\phi^* & -\tau^* \\ ip & -\eta & \eta^* & -iq \end{pmatrix} \tag{65}$$

where Δ, p, and q are real, and p, $q \geqslant 0$. The first term in (65) is a Hamiltonian-type Liouvillian of the form (56), with Δ and θ related as follows to the matrix elements of H in the $\{|\alpha_n\rangle\}$ representation:

$$\Delta = H_{11} - H_{22}, \qquad \theta = -H_{12} \tag{66}$$

Again we note as above under (64) that the Hamiltonian part of $\mathscr{L}$ does not determine H completely, since not all matrix elements of H enter independently into the commutator $[H, \rho]$.

By starting from the form (65) and demanding that the conditions (5) be fulfilled, straightforward but lengthy computation using (6) and (7) leads finally to these necessary and sufficient conditions for the $\mathscr{L}$ given by (65) to generate a dynamical evolution:

$$\begin{aligned}
&(1/i)\{ab^* \mid a\mid^2(\epsilon^* - \eta) + a^*b \mid a \mid^2(\eta^* - \epsilon) + \mid a \mid^4 ip \\
&\quad + ab^* \mid b \mid^2(\eta + \tau^*) + a^*b \mid b \mid^2(-\eta^* - \tau) + \mid b \mid^4 iq \\
&\quad + \mid a \mid^2 \mid b \mid^2[-i(p+q) - (\phi - \phi^*)] + (a^*b)^2 \lambda^* - (ab^*)^2 \lambda\} \geqslant 0
\end{aligned} \tag{67}$$

$$\begin{aligned}
&(1/i)\{ab^* \mid a \mid^2(-\eta - \tau^*) + a^*b \mid a \mid^2(\eta^* + \tau) + \mid a \mid^4 iq \\
&\quad + ab^* \mid b \mid^2(-\epsilon^* + \eta) + a^*b \mid b \mid^2(\epsilon - \eta^*) + \mid b \mid^4 ip \\
&\quad + \mid a \mid^2 \mid b \mid^2[-i(p+q) - (\phi - \phi^*)] + (a^*b)^2 \lambda^* - (ab^*)^2 \lambda\} \geqslant 0
\end{aligned}$$

for every a, b such that $|a|^2 + |b|^2 = 1$. Note that the only parameters in (65) that occur in (67) are those associated with the non-Hamiltonian second term of (65).

7. COMPLETELY NON-HAMILTONIAN STATE EVOLUTION

When the second term in (63) or (65) vanishes, the remaining (Hamiltonian) part of $\mathscr{L}$ generates via (1) the familiar unitary motion of traditional quantum mechanics. Since our present objective is to investigate other possibilities, it is of interest to consider the nature of the motion generated by $\mathscr{L}$ when the first term in (63) or (65) vanishes. Such *completely non-Hamiltonian* evolution is markedly different from the periodic motion that is characteristic of two-level Hamiltonian systems.

Without loss of generality we may assume that the rotation (19)–(21) has been performed so that our completely non-Hamiltonian $\mathscr{L}$ now has, in the quorum $\{\nu_\alpha\}$, the matrix elements

$$(\mathscr{L}_{\beta\alpha}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -C_1 & K_1 & 0 & 0 \\ -C_2 & 0 & K_2 & 0 \\ -C_3 & 0 & 0 & K_3 \end{pmatrix} \tag{68}$$

Combining (68) with the fundamental law of motion (1), we obtain three uncoupled linear differential equations:

$$\dot{s}_n(t) = -C_n + K_n s_n(t), \qquad n = 1, 2, 3 \tag{69}$$

where $\{s_n(t)\}$, the coordinates of the point in $\mathscr{S}$, give the corresponding statistical operator through the relation

$$\rho(t) = (1/\sqrt{2})[\nu_0 + \mathbf{s}(t) \cdot \boldsymbol{\nu}] \tag{70}$$

Now, according to the theorem in Section 5, each $K_n \leqslant 0$; and each $C_n = 0$ unless each K_n is nonzero, in which case the C_n and K_n must characterize a critical ellipsoid with no points external to the unit sphere in $\mathscr{S}$. We consider first the simpler case where at least one of the K_n vanishes. Then (69) becomes

$$\dot{s}_n(t) = K_n s_n(t) \tag{71}$$

and the solution is

$$s_n(t) = s_n(0)\, e^{K_n t} \tag{72}$$

Thus for $K_f = 0$ the corresponding component s_f of $\mathbf{s}$ remains stationary at its initial value, while for $K_m < 0$, s_m exponentially approaches zero. Unlike Hamiltonian evolution, this motion is not cyclic; moreover, as $t \to \infty$, there is in $\mathscr{S}$ a "destination" point, some of whose coordinates (those for which $K_f = 0$) are determined by the initial quantum state.

Next we suppose that each $K_n < 0$ and that the C_n are such that the geometrical criterion (32) is satisfied. The solution to (69) is in this case

$$s_n(t) = \frac{C_n}{K_n} + \left[s_n(0) - \frac{C_n}{K_n}\right] e^{K_n t} \tag{73}$$

Again we find that the motion has a destination, viz.,

$$\lim_{t\to\infty} s_n(t) = \frac{C_n}{K_n} \tag{74}$$

but now that destination is *independent* of the initial conditions. Interestingly, this destination point is on the critical ellipsoid at the intersection of that ellipsoid with the straight line determined by the center of the ellipsoid and the origin of $\mathscr{S}$, as may be readily inferred from (28) and (73).

Such completely non-Hamiltonian motion is of course by itself of no great physical interest. However, its nonperiodic nature and its peculiar destination points are clear indications that there exist superoperators $\mathscr{L}$ in which *both* terms of (63) or (65) are nonzero and which do in fact generate motions of physical interest. In particular, there exist such generalized Liouvillians that are capable of describing energy-conserving but entropy-increasing evolutions of the statistical operator. In the sequel (part III of this series) we shall study in detail, for a two-level system, this physically important class of non-Hamiltonian motions.

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