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Quantum Assembly Semantics: The Fallacious Lingo of Occupation Numbers

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The usual heuristic description of quantum mechanical assemblies features so-called "occupation numbers" interpreted quite literally. This essay critically compares that point of view with a more rigorous understanding of composite systems based upon a principal lesson of Einstein's paradox and Bell's inequality, viz., that it is fallacious to regard a subsystem as possessing or "occupying" any state whatever.

1. MERE SEMANTICS

A remarkable irony in the history of quantum mechanics has been the metamorphosis of the Einstein–Podolsky–Rosen paradox from its original detractive purpose, exposing the unreasonableness of quantum theory, into its currently popular supportive role, illustrating *par excellence* the amazing holistic interconnectedness of the quantal universe. The chief catalyst for this transformation was the theoretical work of J. S. Bell, to whom the present *Foundations of Physics* festschrift series, including this essay, is dedicated.

Bell's quantitative comparison of the predictions of local hidden variables theories with those of quantum mechanics enabled the EPR problem to become experimental, with the celebrated consequence that quantum mechanics—despite all its strangeness, including even those EPR multiple correlations across spacelike intervals that once seemed fantastic—has prevailed and is today generally accepted as the correct fundamental theory of nature.

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Nevertheless, this acceptance, as expressed in textbooks and articles, often seems grudging or halfhearted, more practical then intuitive; indeed physical situations are still commonly described in rather neoclassical terms even when associated calculations are performed in the most modern quantal formalism. For example, one reads elaborate "physical" explanations wherein single electronic states in atoms or solids are said to be "occupied" or not, while attendant computations invoke mathematical structure incompatible with such language.

To construct an apt historical analogy, we might imagine a celestial mechanician in the 18th Century who describes planetary motion "physically" in Ptolemaic terms but faithfully uses Newtonian mathematics to generate his predictions. Such a scholar could justifiably be accused of taking more interest in technical results than in understanding the universe.

In what follows we shall compare critically the usual heuristic description of quantum assemblies of identical particles with the rigorous quantum mechanical treatment of such systems. We shall find that a literal reading of what is ordinarily regarded as the physically intuitive picture of quantum assemblies would in fact lead eventually to theoretical conflict, rather like what might be expected from our 18th-Century scholar if he did not at some point interrupt his Ptolemaic musings with Newtonian computations.

Some readers will undoubtedly be inclined to dismiss the whole analysis as pedantic or perhaps merely semantic. After all, everybody does *use* quantum mechanics to make calculations even when accompanying descriptive terminology seems not purely quantal. Given the meticulousness of what follows, perhaps there is a bit of pedantry. And since we are dealing ultimately with the interpretation—the physical *meaning*—of quantal formalism, there is most assuredly a semantical component to our discussion. It is not, however, "mere" semantics; for the quest to comprehend nature with a new intuition rooted deeply in quantum mechanics rather than some hoary neoclassical paradigm is surely not "mere" in any sense whatever.

2. STATES OF ASSEMBLY AND SUBASSEMBLY

Let $\mathscr{H}(n)$ and $H(n)$ denote respectively the Hilbert space and energy operator for the nth particle in an assembly of N identical particles. Assuming for convenience that $H(n)$ is entirely nondegenerate, we may write

$$H(n) = \sum_{\varepsilon} \varepsilon P_{\varphi_\varepsilon(n)} \equiv \sum_{\varepsilon} \varepsilon P_\varepsilon(n) \tag{1}$$

where $P_{\varphi_\varepsilon(n)} \equiv P_\varepsilon(n)$ is the projector onto the eigenvector $\varphi_\varepsilon(n)$ of $H(n)$ belonging to eigenvalue ε. According to the superselection rule for identical particles, the Hilbert space for the assembly will be a subspace of $\mathscr{H}(1) \otimes \cdots \otimes \mathscr{H}(N)$, either the totally antisymmetric $\mathscr{H}^A$ if the identical particles are fermions or the totally symmetric $\mathscr{H}^S$ if they are bosons. Moreover, in both cases only totally symmetric Hermitian operators that map $\mathscr{H}^A$ (or $\mathscr{H}^S$) into itself are to be regarded as physical observables.

From these elementary principles are deduced the standard theories of Fermi and Bose assemblies familiar features of which include the Slater determinant form of a typical basis vector in $\mathscr{H}^A$,

$$\psi^A_{\varepsilon'\varepsilon''\cdots\varepsilon'''} = \frac{1}{\sqrt{N!}} \begin{vmatrix} \varphi_{\varepsilon'}(1) & \varphi_{\varepsilon'}(2) & \cdots & \varphi_{\varepsilon'}(N) \\ \varphi_{\varepsilon''}(1) & \varphi_{\varepsilon''}(2) & & \vdots \\ \vdots & & & \vdots \\ \varphi_{\varepsilon'''}(1) & & & \varphi_{\varepsilon'''}(N) \end{vmatrix} \tag{2}$$

and similar generic permutant forms for basis vectors $\psi^S_{\varepsilon'\varepsilon''\cdots\varepsilon'''}$ in $\mathscr{H}^S$. In both cases, the bookkeeping is normally simplified by introducing a set of nonnegative integers $\{n_\varepsilon\}$, where n_ε is the number of times the row $(\varphi_\varepsilon(1)\,\varphi_\varepsilon(2)\cdots\varphi_\varepsilon(N))$ occurs in a form like (2). Every ψ^A or ψ^S is described by a unique set $\{n_\varepsilon\}$. Conversely, all vectors of the form ψ^A that are compatible with a given set $\{n_\varepsilon\}$ differ at most by a sign; and each set $\{n_\varepsilon\}$ describes a unique ψ^S.

The connection between these rudimentary aspects of quantum statistics and the rather abstruse issues surrounding the EPR paradox lies in the mathematical observation that ψ^A, and many cases of ψ^S, are generalizations of the multiply correlated states commonly employed in EPR discussions. It follows therefore that any fundamental lesson about quantum mechanics that has emerged from the EPR controversy ought to be incorporated into our understanding of quantum assemblies. In particular, we note that the $N=2$ case of (2) is exactly the mathematical form of the state most commonly employed in EPR investigations, both theoretical and experimental, all of which demonstrate convincingly that when two particles are prepared in such a state, it makes no sense to pretend that one of the particles actually possesses energy ε' (or is in state $\varphi_{\varepsilon'}$) and that the other one has energy ε'' (or is in state $\varphi_{\varepsilon''}$). It would be tedious but not conceptually difficult to extend this conclusion *mutatis mutandis* to higher values of N.

Consequently, when seriously contemplating the physics of Fermi or Bose assemblies, it makes no sense to say that $n_{\varepsilon'}$ particles have energy ε', $n_{\varepsilon''}$ have energy ε'', etc. Yet this misinterpretation of the integers $\{n_\varepsilon\}$ is so common that they are commonly called "occupation numbers," a nomenclature so entrenched that it is imagined to be quite "physical" by many students and professionals in practical chemistry, solid state physics, and elsewhere.

Given that this usual intuitive description of an assembly is foundationally incorrect, the question naturally arises as to what, if anything, can properly be said about, say, a typical particle drawn at random from a Fermi assembly prepared in state (2). The remainder of this essay revolves about this question and related semantical matters, but since none of the points to be made would be qualitatively different for $N>2$, we shall henceforth take $N=2$. Thus simplified, our first problem becomes essentially this: determine the state of a typical fermion in the pair of identical fermions that has total state

$$\psi^A_{\varepsilon'\varepsilon''} = \frac{1}{\sqrt{2}} [\varphi_{\varepsilon'}(1) \otimes \varphi_{\varepsilon''}(2) - \varphi_{\varepsilon''}(1) \otimes \varphi_{\varepsilon'}(2)] \tag{3}$$

In the case of distinguishable particles, the solution would be immediately provided by the theory of reduced density operators. Observables associated exclusively with particle 1 would have the form $A(1) \otimes \mathbf{1}(2)$, and the requirement, for all $A(1)$, that

$$\mathrm{Tr}_1(\sigma(1)\, A(1)) = \langle \psi^A_{\varepsilon'\varepsilon''} |\, A(1) \otimes \mathbf{1}(2)\, | \psi^A_{\varepsilon'\varepsilon''} \rangle \tag{4}$$

would yield as the state of particle 1 the reduced density operator

$$\sigma(1) = \mathrm{Tr}_2 P_{\psi^A_{\varepsilon'\varepsilon''}} = \tfrac{1}{2}(P_{\varepsilon'}(1) + P_{\varepsilon''}(1)) \tag{5}$$

where Tr_n denotes the (partial) trace over Hilbert space $\mathscr{H}(n)$. Parallel results would follow for particle 2.

However, here in the realm of identical particles it is technically incorrect to regard the form $A(1) \otimes \mathbf{1}(2)$ as an observable at all, since the superselection rule demands that all observables be totally symmetric. Accordingly, even if (5) is in some sense correct even for an assembly of identical particles, its derivation must be rather different.

To approach this problem without compromising the notion of indistinguishability, consider the nondegenerate Hermitian operator $A(n)$ defined on $\mathscr{H}(n)$:

$$A(n) = \sum_a a P_{\alpha_a(n)} \equiv \sum_a a P_a(n) \tag{6}$$

A legitimate (i.e., symmetric) observable for the pair of identical particles is the proposition defined by the following procedure: measure A for both particles and ask if one yielded a', the other a''. The operator representing this observable is

$$Q(a', a'') = P_{a'}(1) \otimes P_{a''}(2) + P_{a''}(1) \otimes P_{a'}(2) \tag{7}$$

From $Q(a', a'')$ we can obtain another legitimate assembly observable that seems to refer to a single particle (though not a particular single particle) by introducing this proposition: measure A for both particles and ask if (either) one yielded a. The operator corresponding to this question is

$$Q(a) = P_a(1) \otimes \mathbf{1}(2) + \mathbf{1}(1) \otimes P_a(2) \tag{8}$$

Suppose now that a data set for single fermions is generated by repetition of this procedure:

(i) Prepare a 2-fermion assembly in state (3).
(ii) Select at random one of the fermions.
(iii) Measure A on the fermion selected, and record the result a.

If this is done for many different single-fermion observables A, whatever single-fermion density operator σ correctly summarizes the resultant mean values of those observables would be the most reasonable candidate for "state" of a typical particle in the original assembly of identical particles.

Let w_a be the conditional probability that the particle chosen in step (ii) will yield result a upon measurement of A, given that it were known that one (but not of course which one) would definitely yield a. The probability that measurements of A upon the constituents of a Fermi assembly in state $\psi^A_{\varepsilon'\varepsilon''}$ would yield for one particle the value a is just $\langle \psi^A_{\varepsilon'\varepsilon''} | \, Q(a) \, | \psi^A_{\varepsilon'\varepsilon''} \rangle$. For a single fermion in state $\sigma(n)$, the probability that measurement of $A(n)$ will yield a is of course $\mathrm{Tr}_n(\sigma(n)\, P_a(n))$. Combining all of these observations, we obtain as a necessary condition on the unknown state $\sigma(n)$

$$\mathrm{Tr}_n[\sigma(n)\, P_a(n)] = w_a \, \mathrm{Tr}[P_{\psi^A_{\varepsilon'\varepsilon''}} Q(a)] \tag{9}$$

After substituting (8), the r.h.s. of (9) becomes

$$\begin{aligned} & w_a\{\mathrm{Tr}[P_{\psi^A_{\varepsilon'\varepsilon''}} P_a(1) \otimes \mathbf{1}(2)] + \mathrm{Tr}[P_{\psi^A_{\varepsilon'\varepsilon''}} \mathbf{1}(1) \otimes P_a(2)]\} \\ &\quad = w_a\{\mathrm{Tr}_1[(\mathrm{Tr}_2 P_{\psi^A_{\varepsilon'\varepsilon''}})\, P_a(1)] + \mathrm{Tr}_2[(\mathrm{Tr}_1 P_{\psi^A_{\varepsilon'\varepsilon''}})\, P_a(2)]\} \\ &\quad = w_a\{2\, \mathrm{Tr}_n[\tfrac{1}{2}(P_{\varepsilon'}(n) + P_{\varepsilon''}(n))\, P_a(n)]\} \end{aligned} \tag{10}$$

where the last step follows from the identity of the two particles. Given the arbitrariness of A, combining (9) and (10) implies that

$$\sigma(n) = w_a(P_{\varepsilon'}(n) + P_{\varepsilon''}(n)) \tag{11}$$

our candidate for the single-particle density operator representing the state of a particle randomly selected from the assembly in state $\psi^A_{\varepsilon'\varepsilon''}$. Note that because of the identity of the two particles the form of $\sigma(n)$ is, as would be expected, independent of n. Finally, since a density operator must have trace unity, we find upon taking the trace of both sides of (11) that

$$\sigma(n) = \tfrac{1}{2}(P_{\varepsilon'}(n) + P_{\varepsilon''}(n)) \tag{12}$$

which is in fact the same result noted earlier for the case of distinguishable particles.

We conclude that when a 2-identical-fermion assembly is in state $\psi^A_{\varepsilon'\varepsilon''}$, it is nonsense to believe, as "occupation number" lingo would suggest, that one fermion has energy ε' and the other energy ε''; however, if one wants to attribute a state to the typical fermion, then that state should be the mixture (12). Parallel results hold for similar Bose assemblies and for situations where $N > 2$.

3. ASSEMBLY PHYSICS: HEURISTIC VS. RIGOROUS

Our derivation of (12) was quantally *rigorous* in the important sense that the constituent particles of the assembly were not regarded as possessing energy values, known or unknown; i.e., "occupation numbers" were not taken literally. Nevertheless, it must be admitted that had we followed the more common neoclassical, *heuristic* mode of reasoning about Fermi assemblies, that derivation certainly would have seemed simpler. Indeed all that would have been required is (i) to infer (incorrectly) from the occupation number description of $\psi^A_{\varepsilon'\varepsilon''}$ that one particle has energy ε', the other ε'', (ii) to deduce from the structure of $\psi^A_{\varepsilon'\varepsilon''}$ or perhaps from an indistinguishability argument that a randomly selected particle is equally likely to be the one with ε' or ε'', and (iii) to express this state of ignorance information-theoretically through the artifact of a density operator whose form would obviously be that of (12).

To pursue more deeply the comparison of heuristic and rigorous approaches to quantum assemblies, we need a more precise mathematical account of the heuristic device of taking occupation numbers literally. This is readily obtained. If one fermion actually has energy ε' and the other actually has energy ε'', then the pair is actually either in the state

$\varphi_{\varepsilon'}(1)\otimes\varphi_{\varepsilon''}(2)$ or in the state $\varphi_{\varepsilon''}(1)\otimes\varphi_{\varepsilon'}(2)$; assuming each is equally probable, we conclude that our knowledge of the 2-identical-fermion assembly, in this *heuristic* model, is fully expressed by the assembly density operator

$$\begin{aligned}\rho_h' &= \tfrac{1}{2}[P_{\varepsilon'}(1)\otimes P_{\varepsilon''}(2) + P_{\varepsilon''}(1)\otimes P_{\varepsilon'}(2)] \\ &\equiv \tfrac{1}{2}(P_{\varepsilon'\varepsilon''} + P_{\varepsilon''\varepsilon'}) \end{aligned} \tag{13}$$

Now, it is obvious that ρ_h' is not the same as $\rho_r' = P_{\psi^A_{\varepsilon'\varepsilon''}}$, the density operator that every physicist would surely use in any *rigorous* calculation involving this assembly. Is ρ_h' therefore just a straw man concocted here for the sake of pedantry? I think not; for ρ_h' expresses mathematically what is commonly *said* of such assemblies in heuristic arguments that purport to offer "physical" insight. And ρ_h' thus expresses what many physicists *believe*, or want to believe, about the physical nature of things. Like our fictitious 18th Century celestial mechanician (Sec. 1) who intuits Ptolemaically and computes as a Newtonian, modern scientists often contemplate and discuss quantum assemblies as though descriptions like ρ_h' were correct and relegate true quantum states like ρ_r' to the purely formal, computational attic of mathematical physics.

Despite the fact that ρ_h' and ρ_r' are unequal, they do in certain important cases provide identical physical predictions, a circumstance that tends to support the literal usage of "occupation numbers" for intuitive reasoning. For example, consider a 2-identical-fermion assembly as a member of a canonical ensemble with inverse temperature β. The rigorous canonical density operator has the form

$$\rho_r^A = \sum_{\varepsilon' < \varepsilon''} \frac{e^{-\beta(\varepsilon' + \varepsilon'')}}{Z^A} P_{\psi^A_{\varepsilon'\varepsilon''}} \tag{14}$$

where the sum is restricted in accordance with the antisymmetry inherent in $\psi^A_{\varepsilon'\varepsilon''}$ (Pauli exclusion), and

$$Z^A = \sum_{\varepsilon' < \varepsilon''} e^{-\beta(\varepsilon' + \varepsilon'')} \tag{15}$$

Taking the "occupation number" artifact literally, we may construct an heuristic canonical density operator by using ρ_h', Pauli exclusion, and Boltzmann factors as follows:

$$\rho_h^A = \sum_{\varepsilon' < \varepsilon''} \frac{e^{-\beta(\varepsilon' + \varepsilon'')}}{Z^A} \left[\frac{1}{2}(P_{\varepsilon'\varepsilon''} + P_{\varepsilon''\varepsilon'})\right] \tag{16}$$

Again I stress that (16) captures mathematically what many physicists say and think intuitively, not what they do computationally.

Often in statistical thermodynamics, one only wants to find assembly mean energy U as a function of temperature and extensive parameters such as volume. In this instance both ρ_r^A and ρ_h^A happen to provide the same value for U, viz.,

$$U^A = \mathrm{Tr}(\rho_r^A H) = \mathrm{Tr}(\rho_h^A H) = \sum_{\varepsilon' < \varepsilon''} \frac{e^{-\beta(\varepsilon' + \varepsilon'')}}{Z^A} (\varepsilon' + \varepsilon'')$$

$$= \sum_{\varepsilon' \neq \varepsilon''} \frac{\varepsilon^{-\beta(\varepsilon' + \varepsilon'')}}{2Z^A} (\varepsilon' + \varepsilon'') \tag{17}$$

where $H = H(1) \otimes \mathbf{1}(2) + \mathbf{1}(1) \otimes H(2)$. Can therefore a practical thermodynamicist safely ignore the subtle lessons .of the EPR paradox, consistently adhere to the heuristic model of assemblies, and regard questions as to whether occupation numbers are physically meaningful as "merely semantic"?

To see that the answer to this question is negative, it suffices to compare the entropy values of the 2-identical-fermion assembly as calculated rigorously and heuristically. The rigorous version is well known:

$$S_r^A = -k \,\mathrm{Tr}\, \rho_r^A \ln \rho_r^A = k\beta U^A + k \ln Z^A \tag{18}$$

To perform the heuristic computation, we first rewrite (16) in a simpler spectral expansion:

$$\rho_h^A = \sum_{\varepsilon' < \varepsilon''} \frac{e^{-\beta(\varepsilon' + \varepsilon'')}}{2Z^A} P_{\varepsilon'\varepsilon''} + \sum_{\varepsilon' < \varepsilon''} \frac{e^{-\beta(\varepsilon' + \varepsilon'')}}{2Z^A} P_{\varepsilon''\varepsilon'}$$

$$= \sum_{\varepsilon' \neq \varepsilon''} \frac{e^{-\beta(\varepsilon' + \varepsilon'')}}{2Z^A} P_{\varepsilon'\varepsilon''} \tag{19}$$

Then

$$S_h^A = -k \,\mathrm{Tr}\, \rho_h^A \ln \rho_h^A$$

$$= -k \sum_{\varepsilon' \neq \varepsilon''} [-\beta(\varepsilon' + \varepsilon'') - \ln(2Z^A)] \frac{e^{-\beta(\varepsilon' + \varepsilon'')}}{2Z^A}$$

$$= k\beta U^A + k \ln(2Z^A)$$

$$= S_r^A + k \ln 2 \tag{20}$$

Thus a statistical thermodynamicist who really believed in occupation numbers, who really believed that each fermion actually possessed an energy value, would assign too much entropy to the Fermi assembly.

Similar but quantitatively different results hold for an assembly of identical bosons. Again taking $N = 2$ for convenience, we use totally symmetric basis vectors of the forms

$$\psi^S_{\varepsilon''\varepsilon''} = \frac{1}{\sqrt{2}}[\varphi_{\varepsilon'}(1)\otimes\varphi_{\varepsilon''}(2) + \varphi_{\varepsilon''}(1)\otimes\varphi_{\varepsilon'}(2)], \qquad \varepsilon' < \varepsilon''$$

and

$$\psi^S_{\varepsilon'\varepsilon'} = \varphi_{\varepsilon'}(1)\otimes\varphi_{\varepsilon'}(2) \tag{21}$$

to construct the rigorous canonical density operator

$$\rho^S_r = \sum_{\varepsilon'\leqslant\varepsilon''} \frac{e^{-\beta(\varepsilon'+\varepsilon'')}}{Z^S} P_{\psi^S_{\varepsilon'\varepsilon''}} \tag{22}$$

where

$$Z^S = \sum_{\varepsilon'\leqslant\varepsilon''} e^{-\beta(\varepsilon'+\varepsilon'')} = \sum_{\varepsilon'\neq\varepsilon''} \frac{e^{-\beta(\varepsilon'+\varepsilon'')}}{2} + \sum_{\varepsilon} e^{-\beta(2\varepsilon)} \tag{23}$$

The heuristic model of a 2-identical-boson assembly is derived by an argument entirely analogous to the fermion case, except of course that no Pauli exclusion is invoked when enumerating possible values of the energies possessed or "occupied" by the bosons. Consequently the heuristic canonical density operator would have the form

$$\rho^S_h = \sum_{\varepsilon'\leqslant\varepsilon''} \frac{e^{-\beta(\varepsilon'+\varepsilon'')}}{Z^S}\left[\frac{1}{2}(P_{\varepsilon'\varepsilon''} + P_{\varepsilon''\varepsilon'})\right] \tag{24}$$

As in the fermion case, both rigorous and heuristic computations of assembly energy yield the same result

$$\begin{aligned} U^S = \mathrm{Tr}(\rho^S_r H) = \mathrm{Tr}(\rho^S_h H) &= \sum_{\varepsilon'\leqslant\varepsilon''} \frac{e^{-\beta(\varepsilon'+\varepsilon'')}}{Z^S}(\varepsilon'+\varepsilon'') \\ &= \sum_{\varepsilon'\neq\varepsilon''} \frac{\varepsilon^{-\beta(\varepsilon'+\varepsilon'')}}{2Z^S}(\varepsilon'+\varepsilon'') + \sum_{\varepsilon} \frac{\varepsilon^{-\beta(2\varepsilon)}}{Z^S}(2\varepsilon) \end{aligned} \tag{25}$$

However, once again the calculation of entropy drives a wedge between the two versions. The rigorous entropy has the standard form

$$S^S_r = -k\,\mathrm{Tr}\,\rho_r \ln \rho_r = k\beta U^S + k \ln Z^S \tag{26}$$

To prepare for calculating S_h^S, we first manipulate ρ_h^S as follows:

$$\begin{aligned}\rho_h^S &= \sum_{\varepsilon'<\varepsilon''} \frac{e^{-\beta(\varepsilon'+\varepsilon'')}}{2Z^S}(P_{\varepsilon'\varepsilon''}+P_{\varepsilon''\varepsilon'})+\sum_{\varepsilon}\frac{e^{-\beta(2\varepsilon)}}{Z^S}P_{\varepsilon\varepsilon}\\ &= \sum_{\varepsilon'\neq\varepsilon''} \frac{e^{-\beta(\varepsilon'+\varepsilon'')}}{2Z^S}P_{\varepsilon'\varepsilon''}+\sum_{\varepsilon}\frac{e^{-\beta(2\varepsilon)}}{Z^S}P_{\varepsilon\varepsilon} \qquad (27)\end{aligned}$$

From the spectral form (27), we obtain

$$\begin{aligned}S_h^S &= -k\,\mathrm{Tr}\,\rho_h^S \ln \rho_h^S\\ &= k\beta \sum_{\varepsilon'\neq\varepsilon''} \frac{e^{-\beta(\varepsilon'+\varepsilon'')}}{2Z^S}(\varepsilon'+\varepsilon'')+k\ln(2Z^S)\sum_{\varepsilon'\neq\varepsilon''}\frac{e^{-\beta(\varepsilon'+\varepsilon'')}}{2Z^S}\\ &\quad + k\beta\sum_{\varepsilon}\frac{e^{-\beta(2\varepsilon)}}{Z^S}(2\varepsilon)+k\ln Z^S\sum_{\varepsilon}\frac{e^{-\beta(2\varepsilon)}}{Z^S} \qquad (28)\end{aligned}$$

Comparison of (25) and (28) leads to the form

$$\begin{aligned}S_h^S &= k\beta U+\frac{(k\ln Z^S)}{Z^S}\left(\sum_{\varepsilon'\neq\varepsilon''}\frac{e^{-\beta(\varepsilon'+\varepsilon'')}}{2}+\sum_{\varepsilon}e^{-\beta(2\varepsilon)}\right)\\ &\quad +(k\ln 2)\sum_{\varepsilon'\neq\varepsilon''}\frac{e^{-\beta(\varepsilon'+\varepsilon'')}}{2Z^S} \qquad (29)\end{aligned}$$

which is further simplified by using (23) and (26):

$$\begin{aligned}S_h^S &= S_r^S+(k\ln 2)\sum_{\varepsilon'\neq\varepsilon''}\frac{e^{-\beta(\varepsilon'+\varepsilon'')}}{2Z^S}\\ &= S_r^S+(k\ln 2)\left(1-\sum_{\varepsilon}\frac{e^{-2\beta\varepsilon}}{Z^S}\right) \qquad (30)\end{aligned}$$

Thus, as in the fermion case, faithful adherence to the heuristic approach leads to an erroneously high value for entropy. Similar illustrations, involving larger assemblies, could of course be generated. The point, however, has been adequately made. Our understanding of the "holistic" or "nonlocal" or "nonseparable" character of the quantal universe that has been disclosed by celebrated arguments about the EPR paradox, Bell's inequality, etc. should not be forgotten when we explore even the elementary physics of quantum assemblies.