Foundations of Physics, Vol. 11, Nos. 1/2, 1981

On Completely Positive Maps in Generalized Quantum Dynamics

Ralph F. Simmons, Jr[1,2] and James L. Park[1]

Received April 28, 1980

Several authors have hypothesized that completely positive maps should provide the means for generalizing quantum dynamics. In a critical analysis of that proposal, we show that such maps are incompatible with the standard phenomenological theory of spin relaxation and that the theoretical argument which has been offered as justification for the hypothesis is fallacious.

1. NONUNITARY MOTION IN QUANTUM SYSTEMS AND SUBSYSTEMS

Recently there have been a number of papers arguing that conventional quantum dynamics needs to be generalized so as to include nonunitary evolution of the density operator. Some authors[(1–9)] believe that the need for a new dynamical principle is fundamental, since conventional unitary state evolution in a closed quantum system is incompatible with the second law of thermodynamics. Others[(10–14)] advocate the use of certain nonunitary mappings as elegant phenomenological descriptions of state evolution in subsystems of composite quantum systems which as a whole may still be regarded as undergoing orthodox unitary evolution.

In either case it is obvious that the desired new theory of motion cannot be erected merely on the premise that its maps be not unitary. Some new mathematical concept must be added to quantum theory to give definite structure to any proposed generalization of the dynamical laws. One such proposal that has received some study in depth[(13,14)] involves the concept of complete positivity. The present note offers a critical analysis of that approach.

[1] Department of Physics, Washington State University, Pullman, Washington.
[2] Present address: Disc Memory Division, Hewlett Packard, Boise, Idaho.

0015-9018/81/0200-0047$03.00/0

2. MATHEMATICAL FRAMEWORK

With every quantum system there is associated a complex, separable, complete inner product space, a Hilbert space $\mathscr{H}$. If $\mathscr{H}_A$ and $\mathscr{H}_B$ are Hilbert spaces associated with distinguishable systems A and B, then the direct product space $\mathscr{H}_A \otimes \mathscr{H}_B$ is associated with the composite system of A and B together.

To every reproducible preparation of state for a quantum system there corresponds a density operator ρ, which is a positive-semidefinite, self-adjoint, unit trace linear operator on $\mathscr{H}$. All the density operators are elements of $\mathscr{T}(\mathscr{H})$, the real normed linear space of all linear self-adjoint trace class operators on $\mathscr{H}$. Within the space $\mathscr{T}(\mathscr{H})$ is a set called the positive cone $\mathscr{V}^+(\mathscr{H})$, which contains the positive-semidefinite operators on $\mathscr{H}$, and within the set $\mathscr{V}^+(\mathscr{H})$ is a convex subset $\mathscr{V}_1^+(\mathscr{H})$ containing the elements of $\mathscr{V}^+(\mathscr{H})$ with unit trace. Therefore $\mathscr{V}_1^+(\mathscr{H})$ is the set of density operators, the mathematical representatives of quantum states.

Any proposed dynamical law must involve only mappings of the convex set $\mathscr{V}_1^+(\mathscr{H})$ into itself, and the formulation of such laws is complicated by the fact that neither $\mathscr{V}^+(\mathscr{H})$ nor $\mathscr{V}_1^+(\mathscr{H})$ is a subspace of $\mathscr{T}(\mathscr{H})$. The dynamical postulate of conventional quantum theory describes the motions of a system by a one-parameter (time) unitary group $\{\Lambda_t\}$ of linear transformations of $\mathscr{T}(\mathscr{H})$ into $\mathscr{T}(\mathscr{H})$, the unitary guaranteeing that no density operator will be transformed out of $\mathscr{V}_1^+(\mathscr{H})$. Unfortunately the unitarity also constrains the entropy functional $S = -k \operatorname{Tr} \rho \ln \rho$ to be a constant of the motion.

A natural generalization of the dynamical postulate is obtained by letting $\{\Lambda_t\}$ be a one-parameter semigroup of linear transformations of $\mathscr{T}(\mathscr{H})$ into $\mathscr{T}(\mathscr{H})$ which need not be unitary. In this way irreversible motions are included among the dynamical possibilities, since there may now be maps Λ_t which have no inverse and under which S is not invariant.

The Hille–Yoshida theorem[15] can be used to obtain an equation of motion for ρ in terms of the infinitesimal generator L of the semigroup,

$$\frac{d\rho(t)}{dt} = \frac{d}{dt} \Lambda_t \rho(0) = L\Lambda_t \rho(0) \tag{1}$$

with $\Lambda_t \rho(0) = \rho(t)$,

$$d\rho(t)/dt = L\rho(t) \tag{2}$$

The requirement that mappings $\{\Lambda_t\}$ generated by L transform each element of $\mathscr{V}_1^+(\mathscr{H})$ only into another element of $\mathscr{V}_1^+(\mathscr{H})$ places severe restrictions on L. The conditions satisfied by an admissible L have been given by Kossakowski.[16]

A linear mapping Λ from $\mathscr{T}(\mathscr{H})$ into $\mathscr{T}(\mathscr{H})$ is called positive if $\Lambda\sigma \in \mathscr{V}^{+}(\mathscr{H})$ whenever $\sigma \in \mathscr{V}^{+}(\mathscr{H})$. Thus one approach to generalizing quantum dynamics could be based upon the hypothesis that the equation of motion (2) should feature a generator L associated with positive, trace-preserving maps $\{\Lambda_t\}$. However, this prescription seems rather broad and inexplicit, and some richer, more structured premise is clearly needed.

The particular alternative hypothesis with which the present analysis is concerned involves the notion of complete positivity. A linear transformation Φ from $\mathscr{T}(\mathscr{H})$ into $\mathscr{T}(\mathscr{H})$ is said to be completely positive if the tensor product map $\Phi \otimes I_n$ on $\mathscr{T}(\mathscr{H}) \otimes M(n)$ is positive for all n, where $M(n)$ is the C^*-algebra of $n \times n$ complex matrices.

3. QUANTAL MOTIONS DESCRIBED BY COMPLETELY POSITIVE MAPS

Complete positivity is a stronger restriction than positivity and as such permits one to give the explicit form for the generator of a completely positive dynamical semigroup. Gorini *et al.*[13] have derived the necessary and sufficient conditions for L to be the generator of a completely positive semigroup of an N-level system. As an example, they find the generator for a two-level system, i.e., a system whose associated Hilbert space is two-dimensional. The generator L_g of Gorini *et al.* may be expressed in a matrix representation by using as the basis for $\mathscr{T}(\mathscr{H}_2)$ the matrices

$$\{\nu_i \mid i = 0, 1, 2, 3\} = \left\{\frac{I}{\sqrt{2}} \frac{\boldsymbol{\sigma}}{\sqrt{2}}\right\} \tag{3}$$

where I is the 2×2 identity matrix and $\boldsymbol{\sigma}$ is the standard set of Pauli matrices. The matrix element L_{ij} is defined by

$$L_{ij} = \mathrm{Tr}(\nu_i L \nu_j) \tag{4}$$

In this representation, L_g has the form

$$L_g = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_1 & -\gamma_1 & -h_3 & h_2 \\ a_2 & h_3 & -\gamma_2 & -h_1 \\ a_3 & -h_2 & h_1 & -\gamma_3 \end{pmatrix} \tag{5}$$

where the h_i are related to the energy operator H through

$$H = \sqrt{2} \sum_{i=1}^{3} h_i \nu_i , \qquad \gamma_i \geqslant 0 \tag{6}$$

and

$$a_i = \gamma_i m_i^0 + \sum_{k=1}^{3} \epsilon_{ijk} m_j^0 h_k \tag{7}$$

the m_i^0 being constants subject to conditions which guarantee that $\rho(t)$ is positive.

In this same basis the density operator is conveniently expanded as

$$\rho = \frac{1}{\sqrt{2}} \sum_{i=0}^{3} \rho_i \nu_i \tag{8}$$

so that the equation of motion (2) has the representation

$$\frac{d\rho_i}{dt} = \sum_{j=0}^{3} L_{ij} \rho_j \tag{9}$$

If we interpret the two-level system as a spin-$\frac{1}{2}$ magnetic dipole immersed in a magnetic field **B**, then the energy operator (6) assumes the form

$$H = \sqrt{2} \sum_{i=1}^{3} h_i \nu_i = -\mathbf{m} \cdot \mathbf{B} \tag{10}$$

where **m**, the magnetic moment operator, is given by

$$\mathbf{m} = \tfrac{1}{2}\hbar\alpha\boldsymbol{\sigma} \tag{11}$$

α being the gyromagnetic ratio.

The quantal mean value of the ith component of **m** at time t is computed using the standard trace formula

$$\langle m_i \rangle(t) = \mathrm{Tr}[\rho(t)\, m_i] \tag{12}$$

and the equation of motion for $\langle m_i \rangle$ may therefore be derived from

$$\frac{d\langle m_i \rangle}{dt} = \mathrm{Tr}\left(\frac{d\rho}{dt}\, m_i\right) \tag{13}$$

by using (9) together with a specific choice for L.

If L_g is chosen as the generator, then (5), (6), (9), and (12) yield the equations of motion

$$\frac{d\langle m_i \rangle(t)}{dt} = \sum_{j,k=1}^{3} \epsilon_{ijk} h_j (\langle m_k \rangle(t) - m_k^0) - \gamma_i (\langle m_i \rangle(t) - m_i^0) \tag{14}$$

Gorini *et al.*[13] claim that these equations, a direct consequence of the assumption of complete positivity, are the Bloch equations[17] used in the phenomenological theory of spin relaxation. Were that the case, the hypothesis of completely positive maps would indeed seem to be remarkably fertile, for it would constitute an elegant principle of motion from which at least one well-known phenomenological law could be deduced.

We find, however, that when the standard Bloch equations are written in this same notation, they assume the form

$$\frac{d\langle m_i\rangle(t)}{dt} = \sum_{j,k=1}^{3} \epsilon_{ijk} h_j \langle m_k\rangle(t) - \gamma_i(\langle m_i\rangle(t) - m_i^0) \tag{15}$$

The difference $\langle m_i\rangle(t) - m_i^0$ only appears in the relaxation term in Bloch's equations but appears throughout in Eq. (14) based upon L_g. Therefore, since Gorini *et al.*[13] have shown that L_g satisfies the sufficient and necessary conditions to be a completely positive generator, the evolution as given by the Bloch equations is in fact *not* completely positive. (A more detailed proof of this point is given in the Appendix.) Consequently the hypothesis that completely positive dynamical maps might be useful in describing the quantal motion of subsystems is not supported by the comparison with Bloch's theory.

Neither does it appear that complete positivity will help in the search[1–9] for a new fundamental nonunitary dynamics of closed systems, since the final state $\rho(\alpha)$ of the motion generated by L_g is determined by m_i^0. In fact $\lim_{t\to\infty} \langle m_i\rangle(t) = m_i^0$; and therefore L_g as expressed by (5)–(7) depends on the final state of the system. In our opinion, this feature is philosophically unacceptable in a fundamental theory. A basic dynamical equation should exhibit genuine predictive power, not merely serve as a phenomenological catalog of states of the system. Surely, classical or quantum mechanics would be far less impressive if the generator of time evolution, the Hamiltonian, were itself a function of future states.

It is instructive in this connection to compare the completely positive L_g with the generator derived by Park and Band,[5] who studied all admissible L's for a two-level system in search of those which were independent of ρ and which generated energy-conserving, entropy-increasing motions. Even though complete positivity played no role in their arguments, the generator they found is the same as L_g, provided the first column in (5) vanishes, an essential requirement if L is to be state-independent. One does not obtain the Bloch equations in this case either, and indeed one should hardly expect to derive the Bloch equations without analyzing the composite system of spins plus lattice.

4. IS THERE A VALID PHYSICAL BASIS FOR A COMPLETELY POSITIVE DYNAMICS?

Given the seeming irrelevance of complete positivity in the physical situations discussed above, it is appropriate to inquire whether there exist sound physical arguments in behalf of completely positive dynamical maps. The only published justifications of which we are aware apply only to the motion of subsystems.

Kraus[18] demonstrates that acausal state changes like those associated with measurement intervention in the orthodox quantum theory of measurement are described by completely positive maps, but of course this theorem does not support complete positivity as a basis for generalized causal dynamics.

Gorini *et al.*[13] prove that if the evolution of the isolated system A plus B is given by a unitary group and the constituents A and B are initially uncorrelated, then the subdynamics of system A will be described by a completely positive semigroup. This interesting theorem clearly delineates an important family of motions which are correctly describable by completely positive maps. However, since there exist for isolated composite systems motions which are demonstrably nonunitary,[5] we must conclude that the theorem does not provide strong support for a generalized dynamics based on complete positivity. Moreover, given our earlier result that the Bloch equations are not completely positive, we now have the corollary proposition that the Bloch equations cannot be derived from any model in which the composite system of spins plus lattice is assumed to evolve unitarily from an uncorrelated initial state.

Lindblad's investigation[14] of completely positive maps includes a physical argument for complete positivity which is both more general and more detailed than those discussed above. Two possible interpretations are admitted for the completely positive semigroup evolution of an open subsystem. Either the sybsystem A is interacting with a heat bath or the system A is a subsystem of the isolated system A plus B and the isolated system undergoes unitary time evolution. The argument used by Lindblad to conclude that Λ_t must be completely positive runs as follows.

Consider two quantum systems A and B and a heat bath R. Assume A is interacting with R and that the time evolution of A is given in the Heisenberg picture by the family of maps

$$\Lambda_{A,t}^{*}: \quad \mathscr{B}(\mathscr{H}_A) \to \mathscr{B}(\mathscr{H}_A) \tag{16}$$

where $\mathscr{B}(\mathscr{H}_A)$ is the dual of $\mathscr{T}(\mathscr{H}_A)$, and contains the continuous, linear, self-adjoint operators which represent the quantal observables $\{Q_A\}$ of system

A. [Here we must bear in mind that in the Heisenberg picture ρ_A is constant, and each observable Q_A is represented at different times by different elements of $\mathscr{B}(\mathscr{H}_A)$. Hence the time dependence of mean values such as $\langle Q_A \rangle = \mathrm{Tr}(\rho_A Q_A)$ is induced by mappings (16) in $\mathscr{B}(\mathscr{H}_A)$ which are dual to the mappings $\{\Lambda_{A,t}\}$ in $\mathscr{T}(\mathscr{H}_A)$ considered in previous sections.] Further assume that B is a closed system, its dynamics being governed by the Hamiltonian H_B, and the time evolution of B is given by the conventional unitary mapping, $U_{B,t} = \exp\{itH_B\}$. Let $H_B = 0$ so that $U_{B,t} = I_B$, the identity on $\mathscr{H}_B$.

One now asks if the map $\Lambda_{A,t}^*$ can be extended to a positive map $\Lambda_t^*: \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$, where $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$, such that B is unaffected. It turns out that the map $\Lambda_t^* = \Lambda_{A,t}^* \otimes I_B$ is positive for any N-level system B with $H_B = 0$ if and only if $\Lambda_{A,t}^*$ is completely positive. Hence we are led to believe that the restrictions on the dynamics of A interacting with a heat bath R are in part due to the system B which is not interacting with either A or R. Furthermore, complete positivity follows only if B is an N-level system with arbitrary $N \in \{0, 1, 2, \ldots\}$ and $H_B = 0$.

This physical argument, carefully stated by Lindblad, is clearly based on the requirement that the dynamics of a composite system must be positive even if the composite system is made up of two noninteracting, uncorrelated systems, and there is certainly no reason to question this reasonable requirement. However, to require that the dynamical map $\Lambda_{A,t}^*$ of system A can be extended to a positive map Λ_t^* on $\mathscr{B}(\mathscr{H}_A \otimes \mathscr{H}_B)$ goes beyond the positivity condition. If we assume that $\Lambda_{A,t}^*$ must be such that it can be extended to a positive mapping on $\mathscr{B}(\mathscr{H}_A \otimes \mathscr{H}_B)$, we can show that $\Lambda_{A,t}^*$ must satisfy a positivity requirement in general different from complete positivity. To accomplish this, we simply take system B to be any system which is not an N-level system and for which the Hamiltonian is not trivial, $H_B \neq 0$. For example, take B to be a free particle with $H_B = p^2/2m$. Now $\mathscr{B}(\mathscr{H}_B)$ is not the C^*-algebra of complex $N \times N$ matrices and hence the requirement that $\Lambda_{A,t}^*$ be completely positive does not follow. In general $\Lambda_{A,t}^*$ must satisfy some other positivity condition.

The introduction of system B is unnecessary and has no physical basis. Indeed, if A and B are not interacting, then the state of the composite system A plus B, if initially uncorrelated, remains always in the uncorrelated form $\rho_A \otimes \rho_B$ and this operator on $\mathscr{H}_A \otimes \mathscr{H}_B$ is always positive if ρ_A is positive and if ρ_B is positive. Therefore if the evolution of A maintains ρ_A positive and the evolution of B maintains ρ_B positive, then $\rho_{AB} = \rho_A \otimes \rho_B$ is always positive. Furthermore, since A and B are not interacting, the time evolution of A plus B is always given by the Heisenberg evolution operator $\Lambda_t^* = \Lambda_{A,t}^* \otimes \Lambda_{B,t}^*$. We require only that $\Lambda_{A,t}^*$ and $\Lambda_{B,t}^*$ be positive. They need not be completely positive.

We conclude that the completely positive dynamical map, though mathematically intriguing and interesting as a definite proposal for generalizing dynamics, is a construct too restrictive to serve as the basis for new quantal laws of motion.

APPENDIX

In order for the completely positive equations (14) of Gorini *et al.*[13] to be identical to the Bloch equations (15), the symbols m_i^0 in the two equations would have to be unequal. Thus to compare (14) and (15), we replace m_i^0 in (14) by n_i^0 so that (14) becomes

$$\frac{d\langle m_i\rangle(t)}{dt} = \sum_{j,k=1}^{3} \epsilon_{ijk} h_j(\langle m_k\rangle(t) - n_k^0) - \gamma_i(\langle m_i\rangle(t) - n_i^0) \tag{A1}$$

Gorini *et al.*[13] provide necessary and sufficient conditions for (A1) to generate completely positive maps; in particular, condition (vi) in Ref. 13 states that

$$n_i^0 = 0 \qquad \text{if} \quad \gamma_1\gamma_2\gamma_3 = 0 \tag{A2}$$

Now, if we introduce the symbol m_i^0 through

$$\gamma_i m_i^0 = - \sum_{i,j,k=1}^{3} \epsilon_{ijk} h_j n_k^0 + \gamma_i n_i^0 \tag{A3}$$

Eqs. (A1) become superficially identical to the Bloch equations (15). However, the constants m_i^0 in the Bloch equations are the final values of magnetic moment components.

By solving (A3) for $\{n_i^0\}$ in terms of $\{m_i^0\}$ and $\{h_i\}$, we can show that $\{n_i^0\}$ cannot satisfy both (A1) and (A3). Using standard algebraic procedure, we invert (A3) to obtain

$$\begin{aligned} n_1^0 = {} & [m_1^0(\gamma_1\gamma_2\gamma_3 + \gamma_1 h_1^2) - m_2^0(\gamma_2\gamma_3 h_3 + h_1 h_2 \gamma_2) \\ & + m_3^0(\gamma_2\gamma_3 h_2 - h_1 h_3 \gamma_3)] \\ & \times \left[\sum_{i=1}^{3} \gamma_i h_i^2 + \gamma_1\gamma_2\gamma_3 \right]^{-1} \end{aligned} \tag{A4}$$

When the condition (A2) is invoked in the special case $\gamma_1 = 0$, $\gamma_2 = \gamma_3 = \gamma$ (cf. Ref. 13, p. 824), (A4) becomes

$$n_1^0 = \frac{-m_2^0 \gamma h_3 - m_2^0 h_1 h_2 + m_3^0(\gamma h_2 - h_1 h_3)}{h_2^2 + h_3^2} \tag{A5}$$

Since this is not zero, the constants n_i^0 cannot be chosen so that (A1) becomes the Bloch equations while still satisfying the necessary and sufficient complete positivity conditions in Ref. 13. Therefore Bloch evolution is not completely positive.

REFERENCES

1. J. Mehra and E. C. G. Sudarshan, *Nuovo Cimento* **11**, 215 (1972).
2. G. N. Hatsopoulos and E. P. Gyftopoulos, *Found. Phys.* **6**, 15 (1976).
3. G. N. Hatsopoulos and E. P. Gyftopoulos, *Found. Phys.* **6**, 127 (1976).
4. G. N. Hatsopoulos and E. P. Gyftopoulos, *Found. Phys.* **6**, 439 (1976).
5. G. N. Hatsopoulos and E. P. Gyftopoulos, *Found. Phys.* **6**, 561 (1976).
6. J. L. Park and W. Band, *Found. Phys.* **7**, 813 (1977).
7. W. Band and J. L. Park, *Found. Phys.* **8**, 45 (1978).
8. J. L. Park and W. Band, *Found. Phys.* **8**, 239 (1978).
9. J. L. Park and R. F. Simmons, Jr., in *Old and New Questions in Physics, Cosmology, Philosophy, and Theoretical Biology: Essays in Honor of Wolfgang Yourgrau*, A. van der Merwe, ed. (Plenum Press, New York, 1981), to be published.
10. R. S. Ingarden and A. Kossakowski, *Ann. Phys.* **89**, 451 (1975).
11. E. B. Davies, *Commun. Math. Phys.* **39**, 91 (1974).
12. F. Haake, in *Springer Tracts in Modern Physics, Vol. 66* (Springer, Heidelberg, 1966).
13. V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, *J. Math. Phys.* **17**, 821 (1976).
14. G. Lindblad, *Commun. Math. Phys.* **48**, 119 (1976).
15. K. Yoshida, *Functional Analysis*, 4th ed. (Berlin, 1974).
16. A. Kossakowski, *Bull. Acad. Polon. Sci. Math.* **20**, 1021 (1972).
17. F. Bloch, *Phys. Rev.* **70**, 460 (1946).
18. K. Kraus, *Ann. Phys. (N.Y.)* **64**, 311 (1971).

Foundations of Physics, Vol. 12, No. 4, 1982

Remarks on "On Completely Positive Maps in Generalized Quantum Dynamics"

G. A. Raggio[1] and H. Primas[1]

Received May 8, 1981

The assertion by Simmons and Park that the dynamical map associated with the Bloch equations of nuclear magnetic resonance is not completely positive is wrong.

In a recent paper, Simmons and Park[(1)] claim that Bloch's equation describing the time dependence of the macroscopic nuclear polarization under the influence of an external magnetic field cannot be obtained from a completely positive dynamical evolution law. This assertion is incorrect. Furthermore, Simmons and Park's corollary "that the Bloch equations cannot be derived from any model in which the composite system of spins plus lattice is assumed to evolve unitarily from an uncorrelated initial state" is also wrong (for well-known heuristic derivations compare, e.g., Refs. 2 and 3; for a derivation fulfilling the modern requirements of mathematical rigor compare Ref. 4).

Since Simmons and Park use these false statements to make far-reaching conclusions about the irrelevance of complete positivity for the system-theoretic description of open quantum systems, and since Bloch's system-theoretic description of nuclear magnetic resonance is a paradigm of outstanding significance for modern experimentalists and theoreticians, a direct proof (which is, of course, known) of the complete positivity of the dynamical map associated with Bloch's equation may be appropriate.

The differential equation proposed by Bloch[(5)] for the time dependence

[1] Laboratory for Physical Chemistry, ETH-Zentrum, Zürich, Switzerland.

of the macroscopic nuclear polarization $\mathbf{M}(t) = M_1(e)\mathbf{e}_1 + M_2(t)\mathbf{e}_2 + M_3(t)\mathbf{e}_3$ under the influence of a magnetic field $B\mathbf{e}_3$ is given by

$$\frac{d\mathbf{M}(t)}{dt} = \gamma B\mathbf{M}(t) \times \mathbf{e}_3 - \frac{M_1(t)}{T_2}\mathbf{e}_1 - \frac{M_2(t)}{T_2}\mathbf{e}_2 - \frac{M_3(t) - M_0}{T_1}\mathbf{e}_3$$

where γ is the gyromagnetic ratio, T_1 is the longitudinal and T_2 is the transverse relaxation time fulfilling $2T_1 \geqslant T_2 > 0$. We consider the case of spin 1/2, so that $\mathbf{M}(t)$ is given by the expectation value of $\gamma\mathbf{S}$, where $\mathbf{S} = (S_1, S_2, S_3)$ and S_1, S_2, S_3 are the usual spin-1/2 matrices. A straightforward computation shows that in the Heisenberg representation

$$\mathbf{S}(t) = \exp(tL)\,\mathbf{S}(0)$$

the operator $\gamma\mathbf{S}(t)$ satisfies Bloch's equation for the choice of the generator L as

$$L(A) = i[H, A]_- + \sum_{j=1}^{3} \{V_j{}^*AV_j - \tfrac{1}{2}[V_j{}^*V_j, A]_+\}$$

where A is an arbitrary 2×2 matrix and

$$\begin{aligned} &H = -\gamma BS_3 \\ &V_1 = \alpha_+ S_1 + i\alpha_- S_2, \qquad V_2 = \alpha_+ S_2 - i\alpha_- S_1 \\ &V_3 = \{(2/T_2) - (1/T_1)\}^{1/2} S_3 \\ &\alpha_\pm = \{(\gamma + 2M_0)/4\gamma T_1\}^{1/2} \pm \{(\gamma - 2M_0)/4\gamma T_1\}^{1/2} \end{aligned}$$

The results by Gorini *et al.*[6] and Lindblad[7] imply at once that $\{\exp(tL) \mid t \geqslant 0\}$ is a semigroup of *completely positive* bounded linear maps from the bounded linear operators of a two-dimensional complex Hilbert space into itself. The inequalities $2T_1 \geqslant T_2 > 0$ and $|2M_0/\gamma| \leqslant 1$ are necessary and sufficient for the existence of L as defined above. Hence Bloch's equation for spin 1/2 can be generated by a completely positive dynamical map (compare also Emch and Varilly[8]).

It seems that the more philosophical remarks by Simmons and Park are due to a misunderstanding of the role of system-theoretic descriptions of open quantum systems. Such descriptions never are fundamental, but they have a most useful intermediate position. On the one hand, they allow the experimenter to design and to refine experiments, while on the other hand they are theoretically meaningful and can be rigorously related to the first principles of the theory of matter. Bloch's equations are compatible with the second law of thermodynamics, *and* they can be derived from a larger closed system having an automorphic dynamics. A system-theoretic description of an open system has to be considered as phenomenological; the requirement that it should be derivable from the fundamental automorphic dynamics of a closed system implies that the dynamical map of an open system has to be completely positive. Far from being irrelevant in physical situations, the concept of complete positivity is one of the great new achievements of modern quantum mechanics.

REFERENCES

1. R. F. Simmons and J. L. Park, *Found. Phys.* **11**, 47 (1981).
2. R. K. Wangsness and F. Bloch, *Phys. Rev.* **89**, 728 (1953).
3. A. Abragam, *The Principles of Nuclear Magnetism* (Clarendon Press, Oxford, 1961).
4. J. V. Pulè, *Commun. Math. Phys.* **38**, 241 (1974).
5. F. Bloch, *Phys. Rev.* **70**, 460 (1946).
6. V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, *J. Math. Phys.* **17**, 821 (1976).
7. G. Lindblad, *Commun. Math. Phys.* **48**, 119 (1976).
8. G. G. Emch and J. C. Varilly, *Letters in Math. Phys.* **3**, 113 (1979).

Foundations of Physics, Vol. 12, No. 4, 1982

Another Look at Complete Positivity in Generalized Quantum Dynamics: Reply to Raggio and Primas

Ralph F. Simmons, Jr.[1] **and James L. Park**[2]

Received November 2, 1981

In this rejoinder to a critique by Raggio and Primas of our paper, "On Completely Positive Maps in Generalized Quantum Dynamics," we acknowledge that, contrary to our original assertion, the Bloch equations are indeed completely positive. We then explain briefly why this modification of our analysis does not alter its main conclusions.

As a byproduct of a continuing search for possible generalizations of quantum dynamics that might accommodate in particular the entropy-increasing motions mandated by thermodynamics but forbidden by unitary mechanics, the present authors published last year an analysis[(1)] of completely positive maps. That paper has generated correspondence both praising and condemning its content. Unfortunately our remarks apparently struck some readers as a somewhat inflammatory deprecation of phenomenological studies based on the theory of such maps, even though we explicitly noted that the elegant work[(2)] of Gorini *et al.* "delineates an important family of motions which are correctly describable by completely positive maps." In fact our intent was never to comment adversely or otherwise on the methodology of such fields as nuclear magnetic resonance, but rather to assess the theoretical significance of complete positivity as a possible basis for a generalized quantum thermodynamics.

One section of our paper dealt with the interpretation of a theorem of Gorini *et al.* that provides necessary and sufficient conditions for completely

[1] Disc Memory Division, Hewlett Packard, Boise, Idaho.
[2] Department of Physics, Washington State University, Pullman, Washington.

0015-9018/82/0400-0437$03.00/0

positive maps when the system of interest has a two-dimensional Hilbert space. We proved that these conditions yielded an equation of motion differing from the Bloch equations and then deduced that the latter must not be completely positive. In their recent critique of our article, Raggio and Primas[3] objected to this statement and demonstrated that the mathematical form of the Bloch equations does in fact satisfy the definition of complete positivity; it is then only natural for them to conclude that our entire analysis consequently collapses. The situation is actually a bit more subtle.

First let us acknowledge that Raggio and Primas are correct on the technical point that the Bloch equations are completely positive. In stating the opposite we misinterpreted our own mathematical analysis. In the context of our investigation the problem had been this: Is the concept of complete positivity sufficiently powerful to determine the new laws of motion for general quantum thermodynamics? We were accordingly quite excited by the prospect seemingly offered by the aforementioned theorem of Gorini *et al.*, which we had thought of as a derivation of the Bloch equations from the principle of complete positivity. Further consideration revealed problems with this view, fully discussed in the Appendix of our paper, which contains the mathematics we misinterpreted as establishing that Bloch's equations were not completely positive. The conclusion we should have drawn from this analysis was simply that the principle of complete positivity is not sufficient to determine Bloch's equations. Therefore our claim that comparison of the work of Gorini *et al.* with Bloch's theory does not support"... the hypothesis that completely positive dynamical maps might be useful in describing the quantal motion of subsystems..." remains valid in the context in which we stated it.

If the conditions required by complete positivity were necessary and sufficient to derive the Bloch equations, then complete positivity would offer an excellent foundation for a new, non-Hamiltonian dynamical postulate of the type that we and others[4] are seeking. Unfortunately, the Bloch equations emerge from complete positivity only when a special additional condition ($\mathbf{m}_0 \times \mathbf{h} = 0$) is imposed. In fact it is known that complete positivity also includes ordinary Hamiltonian motion as a special case. Thus, for us, complete positivity is essentially a neutral criterion, favoring neither entropy-conserving nor entropy-increasing forms of mechanics.

Though we regret the misstatement about the Bloch equations and are pleased to clarify the matter, nevertheless our final conclusion is in essence no different: Despite its formal beauty and apparent phenomenological utility, the completely positive map is not an adequate construct for the formulation of fundamental dynamical principles.

REFERENCES

1. R. F. Simmons, Jr. and J. L. Park, *Found. Phys.* **11**, 47 (1981).
2. V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, *J. Math. Phys.* **17**, 821 (1976).
3. G. A. Raggio and H. Primas, *Found. Phys.*, this issue, preceding paper.
4. G. N. Hatsopoulos and E. P. Gyftopoulos, *Found. Phys.* **6**, 15, 127, 439, 561 (1976).

Printed in Belgium