# Relations among elements of the density matrix. II. Exotic conservation laws

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Ordinary quantal conservation laws are associated with null operators for time rate of change and are valid for causal evolution of the system through either pure or mixed states. There is, however, a larger class of quantal constants of the motion, some of which are conserved only for pure state evolution. After an analysis of the theoretical origins of these exotic conserved quantities, several illustrations are presented with empirical interpretations based on the quorum theory methods used in Paper I.

## 1. ORDINARY CONSERVATION PRINCIPLES

The usual procedures for the identification of conserved quantities in quantum physics are based strongly on analogies to elegant classical schemes for obtaining constants of the motion. Thus, the formal parallelism between algebraic properties of Poisson brackets and quantal commutators is often exploited in the search for conserved quantal observables; similarly, Noether's theorem is routinely extended to quantum field theory as a means of finding expressions for conservation laws. However, because these methods rely so fundamentally upon the classical framework, they fail to generate all classes of quantally conserved measurable quantities.

To see why this is the case, it is necessary to recall the rather different relationships between data and theoretical observables that distinguish classical and quantal physics. Consider mechanics. In the classical case, observables are represented as functions of state (phase) and the numerical values of these functions are identified in principle with numerical data. A constant of the motion is then simply a phase function whose total time derivative vanishes, the consequent fixed value of the function being equal to the constant measured value of the observable represented by the function.

In quantum mechanics, on the other hand, observables are represented by Hermitian operators whose relation to data is more indirect. The testable assertions of quantum theory do not refer to "values" of observables, but, rather, to mean values of statistical collectives of data gathered from ensembles of identical experiments. Thus, to say in quantum mechanics that an observable A is "conserved" can mean, in terms of data, nothing more than that $\langle A\rangle_1$, the mean value computed from $A$-data referring to time $t_1$, is equal to $\langle A\rangle_2$, computed from $A$-data associated with $t_2$, where $t_1$ and $t_2$ are arbitrary.

To emphasize the difference between this quantal statement of conservation and that usually implied in classical theory, we shall call the classical version *point-by-point conservation* and the quantum idea *conservation-in-the-mean*. Crudely stated, a point-by-point conservation law asserts that "at every measurement the conserved observable has a definite (unique) value which is independent of time," whereas conservation-in-the-mean requires only that "the mean value of measurement-results on the observable is independent of the time lapse between preparation and measurement".

Since point-by-point conservation is in fact an unphysical concept, at best an abstract idealization from the facts of life in the physical laboratory, it could be argued cogently that conservation-in-the-mean with its realistic statistical statements should be acceptable whether one is using classical or quantum theory. Nevertheless, ordinary quantal conservation theory attempts to mimic its classical counterpart in the following well-known manner.

With every Hermitian operator $A$ there is associated another such operator called the "time rate of change of $A$" (symbol $dA/dt$) and defined by

$$\frac{dA}{dt}=\frac{1}{i\hbar}[A,H]+\frac{\partial A}{\partial t} \tag{1}$$

where $H$ is the Hamiltonian of the system, and $\partial A/\partial t$ denotes the time derivative of the operator $A$, should $A$ be defined with intrinsic time dependence. The physical significance of $dA/dt$ rests on a theorem which establishes that the *time derivative of the mean value of* $A$ (a number empirically obtainable by computation from $A$-data) is numerically equal to the quantally calculated *mean value of the operator* $dA/dt$.

Because the form of (1) is reminiscent of the analogous classical Poisson bracket relation from which the necessary and sufficient condition for the point-by-point conservation of a physical quantity is immediately evident (vanishing of bracket plus intrinsic derivative), conventional quantum mechanics normally declares a conserved observable $A$ to be one for which the operator $dA/dt$ is null. Usually $\partial A/\partial t$ is zero and the criterion for ordinary conservation becomes simply the vanishing of the commutator $[A,H]$.

Since point-by-point conservation is meaningless in quantum physics, the latter standard formulation of quantal conservation theory is overly restrictive. In fact, the vanishing of $dA/dt$ is a sufficient but *not* a necessary condition for $A$ to be conserved-in-the-mean. We shall refer to the mean of an observable which satisfies this *sufficient* condition as an *ordinary* conserved quantity.

Consequently, as we shall demonstrate below, there exist operators $A$, for which $dA/dt$ is not the null operator, but which nevertheless are conserved-in-the-mean. Moreover, we shall find that there exist time-independent nonlinear combinations of several quantal mean values, none of which is individually conserved in any sense. We call such extraordinary quantal constants of the motion *exotic conserved quantities*.

For later reference, one characteristic feature of *ordinary* conserved quantities should be especially noted: If $dA/dt$ vanishes, then $\langle A\rangle$ is time-independent regardless of whether the evolving quantum state is pure or mixed. By contrast, there are exotic conserved quantities which are constant only for pure state evolution.

## 2. EXOTIC CONSERVED QUANTITIES

In Paper I we reviewed the concept of quorum[1] and indicated how elements of the statistical matrix could be expressed as functions of quorum means. Hence, if some algebraic combination of the matrix elements were invariant under temporal evolution, that combination could be physically interpreted using quorum theory and a conserved quantity would thereby be identified as a function of the quorum means.

### A. Conservation of the statistical determinant

The causal evolution of a statistical operator $\rho$ is effected by a unitary evolution operator $U$ determined by the system Hamiltonian; thus

$$\rho(t_2) = U(t_2, t_1)\rho(t_1)U^{\dagger}(t_2, t_1). \tag{2}$$

From (2) and the theory of determinants it follows immediately that

$$\det\rho(t_2) = \det\rho(t_1), \tag{3}$$

i.e., the statistical determinant is a constant of the motion.

Naturally, this theorem will produce an interesting conserved quantity only for finite-dimensional Hilbert spaces.

### B. Pure state conservation of statistical minors

In our study of definiteness inequalities in Paper I, we observed that the principal minor determinants of the statistical matrix can be of special significance. It was noted in particular, that for pure quantum states all such minor determinants of dimension exceeding unity vanish. Now from (2) it is readily shown that $\rho(t_2)$ will be pure if $\rho(t_1)$ was pure; i.e., pure states evolve into pure states, a well-known quantal theorem. Hence *for pure state evolution*, the quorum means occurring in any principal minor must vary in time in such a manner that the minor determinant remains fixed at zero. We have, therefore, a prolific source of measurable quantities conserved during pure state evolution.

The question now arises as to whether the minor determinants are also conserved in the time evolution of mixed states. Investigation shows that while it is possible in specific instances for minor determinants to be conserved for both pure and mixed states, in general only the pure state conservation law holds.

For example, consider a three-dimensional Hilbert space. Let the initial statistical matrix be

$$(\rho(t_1)) = \begin{pmatrix} w & a & b \\ a^* & x & c \\ b^* & c^* & y \end{pmatrix} \tag{4}$$

and let the evolution matrix for the time interval of interest be

$$(U) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5}$$

After substituting (4) and (5) into (2) we obtain

$$(\rho(t_2)) = \begin{pmatrix} w & b & a \\ b^* & y & c^* \\ a^* & c & x \end{pmatrix}. \tag{6}$$

The upper left minor determinant is conserved only if

$$wx - |a|^2 = wy - |b|^2. \tag{7}$$

Certainly (7) is not generally true; it essentially demands the equality of two of the principal minor determinants of $\rho(t_1)$, a necessary condition only if $\rho(t_1)$ is pure.

We conclude that the principal minors of the statistical matrix, when interpreted in terms of quorum means, will provide a family of rather anomalous constants of the motion, always conserved in pure state evolution but not necessarily conserved otherwise. Thus, the pure state definiteness equalities exhibited in Paper I are examples of exotic conservation laws.

### C. Conservation of functions of $\rho$

The mean value of any function of $\rho, F(\rho)$, is a conserved quantity, regardless of whether or not $dF(\rho)/dt$ is null. For example, consider $F(\rho) = \rho^2$:

$$\begin{aligned} \frac{d\rho^2}{dt} &= \frac{1}{i\hbar}[\rho^2, H] + \frac{\partial\rho^2}{\partial t} \\ &= \frac{1}{i\hbar}[\rho^2, H] + 2\rho\frac{\partial\rho}{\partial t}. \end{aligned} \tag{8}$$

According to the quantal Liouville theorem,

$$\frac{\partial\rho}{\partial t} = \frac{1}{i\hbar}[H, \rho]. \tag{9}$$

Hence,

$$\begin{aligned} i\hbar\frac{d\rho^2}{dt} &= \rho^2 H - H\rho^2 + 2\rho H\rho - 2\rho^2 H \\ &= -\rho^2 H + 2\rho H\rho - H\rho^2 = [[\rho, H], \rho] \not\equiv 0. \end{aligned} \tag{10}$$

Thus, $\rho^2$ is not conserved in the ordinary (classically inspired) sense because its associated time rate of change operator fails to vanish. Nevertheless, $\langle\rho^2\rangle$ is a constant of the motion due to a property of the trace

$$\begin{aligned} i\hbar\frac{d}{dt}\langle\rho^2\rangle &= i\hbar\left\langle\frac{d\rho^2}{dt}\right\rangle \\ &= i\hbar\,\mathrm{Tr}\{\rho(-\rho^2 H + 2\rho H\rho - H\rho^2)\}. \end{aligned} \tag{11}$$

Since $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, the right side of (11) is zero, even if $d\rho^2/dt \not\equiv 0$.

Similarly, it can be shown that the mean value of any function of $\rho$ is conserved. A famous case in point is $\ln\rho$, whose mean value is proportional to the entropy in statistical mechanics.

It is possible to relate the conservation of the statistical determinant discussed above to this idea that functions of $\rho$ generate exotic constants of motion. If there exists an operator $D(\rho)$ such that

$$\langle D(\rho)\rangle = \mathrm{Tr}[\rho D(\rho)] = \det\rho, \tag{12}$$

then the conservation of $\det\rho$ could be regarded as a consequence of the fact that $\det\rho$ is the mean value of a function of $\rho$.

In general, many operators $D(\rho)$ can be found which satisfy (12). Let the eigenvalues of $\rho$ be $\{w_k\}$ and of $D(\rho)$ be $\{d_k\}$. Since $\det\rho$ is the product of the eigenvalues of $\rho$, and $\rho$ and $D(\rho)$ are both diagonal in the same matrix representation, (12) may be written as

$$w_1 w_2 \cdots w_N = \sum_{k=1}^{N} w_k d_k, \tag{13}$$

where $N$ is the dimensionality of the Hilbert space.

One solution of (13) is given by

$$d_k = \begin{Bmatrix} w_2 \cdots w_N, & k=1 \\ 0 \quad , & k \neq 1 \end{Bmatrix}. \tag{14}$$

Many other solutions could be obtained similarly by inspection of (13).

## 3. ILLUSTRATIONS

Examples of exotic conserved quantities are presented below in the same format used for definiteness inequalities in Paper I.

### A. Spin-½ system

Quorum: $\sigma_x, \sigma_y, \sigma_z$.

Statistical matrix:

$$(\rho) = \tfrac{1}{2}\begin{pmatrix} 1 + \langle\sigma_z\rangle & \langle\sigma_x\rangle - i\langle\sigma_y\rangle \\ \langle\sigma_x\rangle + i\langle\sigma_y\rangle & 1 - \langle\sigma_z\rangle \end{pmatrix}. \tag{15}$$

Conserved statistical determinant:

$$\det\rho = \tfrac{1}{4}(1 - \langle\sigma_z\rangle^2 - \langle\sigma_x\rangle^2 - \langle\sigma_y\rangle^2). \tag{16}$$

From the time independence of (16) it follows that the quantity

$$\Sigma \equiv \langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 \tag{17}$$

is conserved. It is also possible to establish the constancy of (17) by standard manipulations, starting from the observation that

$$\frac{d(\sigma_x + \sigma_y + \sigma_z)^2}{dt} = 0. \tag{18}$$

### B. Harmonic oscillator with 2-level energy cutoff

(For a short explanation of the concept of cutoff observable,[2] consult Paper I.)

Quorum: $x, p, H = (p^2/2m) + (m\omega^2/2)x^2$.

Statistical matrix: Let $\rho_c$ denote the $2 \times 2$ nonzero submatrix of $\rho$. The cutoff is assumed to occur after the two lowest energy levels:

$$\rho_c = \tfrac{1}{2}\begin{pmatrix} 3 - \langle K\rangle & \langle X\rangle - i\langle P\rangle \\ \langle X\rangle + i\langle P\rangle & -1 + \langle K\rangle \end{pmatrix}, \tag{19}$$

where

$$K \equiv \frac{2}{\hbar\omega}H, X \equiv \left(\frac{2m\omega}{\hbar}\right)^{1/2} x, P \equiv \left(\frac{2}{m\hbar\omega}\right)^{1/2} p. \tag{20}$$

Conserved minor determinant: The submatrix is known to be correct only for times when the cutoff exists. However, since the cutoff is in the *energy* and the energy probability distribution for the oscillator is time-independent, we conclude that (19) is valid at all times.

According to the theory of Sec. 2B, $\det\rho_c$ is conserved for all pure state evolutions. Since $\det\rho_c$ contains the mean value of $H$, which is conserved in the ordinary sense, it follows that the terms in $\det\rho_c$ not containing $\langle H\rangle$ must be separately conserved.

Thus, our theory predicts that the quantity

$$\tfrac{1}{4}(\langle X\rangle^2 + \langle P\rangle^2) = \frac{m\omega}{2\hbar}\langle x\rangle^2 + \frac{1}{2m\hbar\omega}\langle p\rangle^2 = (\hbar\omega)^{-1}\mathcal{E}(\langle x\rangle, \langle p\rangle), \tag{21}$$

with

$$\mathcal{E}(\langle x\rangle, \langle p\rangle) \equiv \frac{\langle p\rangle^2}{2m} + \frac{m\omega^2}{2}\langle x\rangle^2, \tag{22}$$

will be conserved in pure state evolution of the cutoff harmonic oscillator. The final result is not new. It is well known that the harmonic oscillator meets the requirements of Ehrenfest's theorem,[3] hence the classical energy function (22) with quantal means as arguments is conserved for all types of time evolution, including the pure state, cutoff case of the present example. Note that $\mathcal{E}$ is not the same thing as $\langle H\rangle$; there is no single Hermitian operator associated with $\mathcal{E}$ yet it is a meaningful physical quantity.

### C. Harmonic oscillator with 3-level energy cutoff

Quorum: $x, x^2, x^3, p, p^3, xp + px, H, H^2$.

Statistical matrix: Let $\rho_c$ denote the $3 \times 3$ nonzero submatrix of $\rho$. The cutoff is assumed to occur after the three lowest energy levels.

$$\rho_c = \begin{pmatrix} -\tfrac{1}{8}[8\langle K\rangle - \langle K^2\rangle - 15] & \tfrac{1}{2}[2\langle X\rangle - \langle Z\rangle - i(2\langle P\rangle - \langle Q\rangle)] & \tfrac{1}{2}[(1/\sqrt{2})(\langle Y\rangle - \langle K\rangle) - i\langle A\rangle] \\ \tfrac{1}{2}[2\langle X\rangle - \langle Z\rangle + i(2\langle P\rangle - \langle Q\rangle)] & \tfrac{1}{4}[6\langle K\rangle - \langle K^2\rangle - 5] & (1/2\sqrt{2})[\langle Z\rangle - \langle X\rangle - i(\langle Q\rangle - \langle P\rangle)] \\ \tfrac{1}{2}[(1/\sqrt{2}(\langle Y\rangle - \langle K\rangle) + i\langle A\rangle] & (1/2\sqrt{2})[\langle Z\rangle - \langle X\rangle + i(\langle Q\rangle - \langle P\rangle)] & \tfrac{1}{8}[\langle K^2\rangle - 4\langle K\rangle + 3] \end{pmatrix}, \tag{23}$$

where

$$\begin{aligned} &X \equiv (2m\omega/\hbar)^{1/2}x, \quad Y \equiv (2m\omega/\hbar)x^2, \\ &\quad Z \equiv \tfrac{1}{3}(2m\omega/\hbar)^{3/2}x^3, \\ &P \equiv (2/m\hbar\omega)^{1/2}p, \quad A = (1/\hbar\sqrt{2})(xp + px), \\ &\quad Q \equiv \tfrac{1}{3}(2/m\hbar\omega)^{3/2}p^3, \\ &K \equiv (2/\hbar\omega)H, \quad K^2 = [4/(\hbar\omega)^2]H^2. \end{aligned} \tag{24}$$

Conserved minor determinant: Consider the upper left $2 \times 2$ minor of $\rho_c$. The diagonal elements, being functions of $H$, are conserved separately. Hence we may assert that, at least for pure state evolution, the following quantity is a constant of the motion:

$$\Lambda_1 \equiv [2\langle X\rangle - \langle Z\rangle]^2 + [2\langle P\rangle - \langle Q\rangle]^2. \tag{25}$$

Similarly, from the lower right 2 × 2 minor another conserved quantity may be derived:

$$\Lambda_2 \equiv [\langle X\rangle - \langle Z\rangle]^2 + [\langle P\rangle - \langle Q\rangle]^2. \tag{26}$$

Subtracting $\Lambda_2$ from $\Lambda_1$ and simplifying, we get

$$\Lambda_3 = 3(\langle X\rangle^2 + \langle P\rangle^2) - 2(\langle X\rangle\langle Z\rangle + \langle P\rangle\langle Q\rangle), \tag{27}$$

which is of course likewise conserved for pure state evolution. But,

$$\langle X\rangle^2 + \langle P\rangle^2 = 4(\hbar\omega)^{-1}\mathcal{E}(\langle x\rangle, \langle p\rangle), \tag{28}$$

where $\mathcal{E}$ is defined as in (22).

Recalling again that $\mathcal{E}(\langle x\rangle, \langle p\rangle)$ is a constant of the motion because of Ehrenfest's theorem, we conclude that the following combination of quorum means is an exotic conserved quantity at least for pure state evolution

$$\Lambda \equiv \frac{4}{3\hbar^2}(\langle X\rangle\langle Z\rangle + \langle P\rangle\langle Q\rangle) = (m\omega)^2\langle x\rangle\langle x^3\rangle + (m\omega)^{-2}\langle p\rangle\langle p^3\rangle. \tag{29}$$

Note that $\Lambda$ is a nonlinear function of four quorum means and that $\Lambda$ has *not* been obtained by finding an operator $L$ such that

$$\langle L\rangle = \Lambda, \quad \frac{dL}{dt} = 0. \tag{30}$$

Additional exotic conserved quantities for this system could similarly be generated from the remaining two-dimensional minor and from the three dimensional minor determinant ($\det\rho_c$).

### D. Spin-1 system

Several exotic conservation laws may be obtained by calculating the determinant and minor determinants of the statistical matrix given by (20) in Paper I.

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[2]J. L. Park and W. Band, Found. Phys. **1**, 339 (1971).
[3]A. Messiah, *Quantum Mechanics* (Amsterdam, North-Holland, 1961), p. 217.