Nature of Quantum States*

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This paper carefully explores the relations among the statistical ensembles, systems, and states (pure and mixed) of quantum theory. By systematically contrasting the classical and quantal realizations of a general paradigm for a probabilistic physics, important distinctions are exposed both in statics and dynamics. Included are observations concerning the intrinsic ambiguity of the quantum-state concept and the peculiarly quantum property of dynamic indivisibility. It is concluded that the conceptual gulf between classical states and quantum "states" is wider than commonly assumed.

INTRODUCTION

THE dominant theme of the quantum theory—though many textbooks do not emphasize it sufficiently—is that all its *causal* statements are *probabilistic.* In other words, the epistemic rule of correspondence[1] which provides the empirical meaning of quantum-theoretical *states* involves probabilistic concepts in an essential way. This hallmark of quantum theory must be borne in mind constantly, if the physical significance of the theory is to be understood at all. Moreover, it is of fundamental importance to recognize that the probabilistic rule alone is insufficient to link abstract states with empirical experience. Indeed pure probability theory is itself a formidable collection of abstruse constructs requiring further rules of correspondence of its own. The situation is wholly analogous to that prevailing in geometry, where pure geometry is converted to physical geometry by appending familiar operational definitions involving straightedge and compasses. In the case of probability theory, the required connections are implicit in established practices of experimental science. Especially to be noted is the well-known identification of physical probability as relative frequency in a *statistical ensemble.* The appropriateness of the frequency definition of probability is an old philosophic problem, but we are not concerned with it here. We simply accept uncritically the fact that in all scientific applications of probability theory, the construct probability is epistemically linked to statistics.[2] Accordingly, if quantum theory is to be understood as a physical science, it is absolutely essential to remember that its primary connection to the empirical world is through statistical ensembles. To ignore the statistical aspect of quantum theory is to dismiss much of its relevance to actual physical experience.

Nevertheless, it has proved intuitively useful in ordinary quantum-theoretical applications to think of the state vector (or its wavefunction representative) as belonging to a single system at a single time in the same way that states belong to individual systems in classical mechanics. Thus the jargon of modern physics easily induces one to regard the phrase "an electron in state ψ" as merely the quantum analog to the classical expression "an electron in state (q_0, p_0)," in spite of the fact that the former refers physically to statistics of measurement results upon an ensemble of identically prepared electrons[3] whereas the latter just means that a single (classical) electron *has* position q_0 and momentum p_0. Superficially, this common phraseology seems innocent enough; indeed one might be disposed to think that a theoretician could use it unreservedly without contradiction so long as he remembered to switch to the correct statistical

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[1] H. Margenau, *The Nature of Physical Reality* (McGraw-Hill Book Co., New York, 1950).

[2] R. von Mises, *Mathematical Theory of Probability and Statistics* (Academic Press Inc., New York, 1964), pp. 43-49; cf. also H. Margenau, *op. cit.,* Chap. 13.

[3] The quantum ensemble may consist of many independent electrons at one time, one electron sequentially measured and reprepared, or any combination of these two extremes, Cf. H. Margenau, Phil. Sci. 4, 352 (1937).

meaning of ψ at the conclusion of his arguments and calculations.

It is the purpose of the present study to expose certain logical weaknesses inherent in the drawing of structural parallels between classical and quantum physics and hence to demonstrate that a quantum theorist may *not* in every context relate state vectors to single systems in the classical manner described above. The linguistic extension of ψ from its role in describing ensembles to its further function as the state of a single system has given birth to monumental barriers to the understanding of quantum theory as a rational branch of natural philosophy. Problems connected with the general theory of measurement[4]—the nature of quantum measurement, wave-packet reduction, concepts of compatibility and simultaneous measurement—are especially aggravated by this popular convention that the state of an individual system is represented by ψ.

Thus, in our opinion the material to be surveyed below forms the essential prelude to any serious study of the basic philosophic issues associated with quantum theory. We have attempted to show elsewhere[5,6] that problems concerning measurement in quantum physics can be sharpened, and sometimes resolved, by according proper attention to those basic physical characteristics of quantum states with which the present essay deals.

I. MINIMAL AXIOMATIC STRUCTURE OF A PROBABILISTIC PHYSICS

Rather than devising, in the customary way, artificial verbal analogies between the constructs of classical and quantal physics (e.g., the aforementioned state (q_o, p_o) and "state" ψ), we *contrast* the two kinds of theories in a manner which reveals formal differences as readily as the standard comparisons indicate formal similarities. To facilitate such a comparison of classical and quantal statistical physics, it is useful to consider an abstract paradigm theory representative of a probabilistic physical theory in general. As in all physical theories, the primitive idea of this prototype theory is the study of the *numerical results* $\{a_k\}$, $\{b_l\}$, $\cdots$ of measurements of *observables* a, b, $\cdots$ performed on a physical system **S**. It is the goal of the theory to incorporate these measurement results into a *causal* framework, i.e., information about present measurement results should determine similar information about future measurement results. The defining property of the present model theory is the fact that this information has a form related to the *probabilities* of the measurement results. Hence what is actually studied is the *statistics* of the results of measurements on the member systems of *ensembles of identically prepared replicas of the physical system of interest* **S**. In the theory, an ensemble is characterized at a given time by the arithmetical mean values (expectation values) of measurement results for the various observables, i.e., by the "functional" $<a>$ whose "domain" is the set of observables. In such a theory the causal nexus links the values of $<a>$ at different times for every a.

The essential axioms which typify a statistical physical theory may be summarized as follows:

P1: Mathematical objects A, B, $\cdots$ correspond to observables a, b, $\cdots$

P2: For every ensemble of identically prepared systems there exists a functional $m_t(A)$ such that for every pair (A, a),
$$m_t(A) = <a>_t,$$
the arithmetic mean of the results of a measurements performed at time t relative to the preparation of each member system.

P3: For every kind of physical system (i.e., system of interest plus its relevant environment) there exists a causal law relating the forms of the mean value functional at different times.

Later more definite content is ascribed to these rather vague statements. First, however, we must review an important scheme for the classification of statistical ensembles especially emphasized by

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[4] The extensive literature of quantum measurement theory includes these papers published in this JOURNAL: E. P. Wigner, Am. J. Phys. 31, 6 (1963); A. Shimony, Am. J. Phys. 31, 755 (1963).

[5] J. L. Park, "Quantum Theoretical Concepts of Measurement," Part II, Thesis, Yale University (1967).

[6] J. L. Park and H. Margenau, "Simultaneous Measurability in Quantum Theory" (to be published).

von Neumann.[7] This classification hinges on the concept of ensemble homogeneity, a property which ultimately depends on the mathematical character of measurement statistics associated with the ensemble. In particular, it is always possible to conceive of many subdivisions of a given ensemble into subensembles; these subensembles must themselves be bona fide statistical ensembles, i.e., they must each contain an "effectively infinite" number of systems so that the frequency definition of probability may still be used. The homogeneity of the original ensemble is determined by comparing the statistical characteristics of such subensembles. This process of subdivision is of course a *mental* operation based on statistics of measurement results; indeed it should always be remembered that the ensemble concept includes even the case of a single system in a temporal sequence of measurements and repreparations. Most ensembles are *mixed,* in the sense that the subensembles into which they may be grouped are statistically distinct; there exist, however, completely homogeneous, or *pure,* ensembles for which every subdivision yields subensembles all statistically identical to the original. Mathematically, a mixed ensemble is characterized by a mean value functional $m(A)$ such that there exist distinct functionals $m_1(A)$, $m_2(A)$ in terms of which $m(A)$ may be expressed as follows:

$$m(A) = w_1 m_1(A) + w_2 m_2(A), w_1, w_2 > 0,$$

where w_1, w_2 denote the respective fractions of the systems in the original ensemble present in subensembles 1 and 2. (Clearly, $w_1 + w_2 = 1$.) If, on the other hand, for a given $m(A)$ there do not exist distinct $m_1(A), m_2(A)$ such that $m(A) = w_1 m_1(A) + w_2 m_2(A)$, the ensemble characterized by that $m(A)$ is of the pure, or homogeneous type. As we see later, it is the latter type of ensemble which, due to its maximal order and uniformity, may in some cases provide a means to extract from an initially probabilistic theory (one dealing basically with ensembles and statistics) a plausible *state* concept applicable to single systems in the classical sense, i.e., a state concept such that any physical system considered at a given instant may be regarded without contradiction as *having* a definite state at that instant.

To exhibit a simple realization of the general paradigm above and to exemplify the concept of ensemble homogeneity, it is instructive to examine classical statistical mechanics from an unusual perspective which disregards the original purpose of that classical theory as a mechanical explanation of thermodynamics. That is, we are not interested in the Gibbsian *imaginary* ensemble of replicas and do not adopt the usual postulate which connects averages over that ensemble with thermodynamic parameters associated with the single system of interest. Instead, we consider a real ensemble (either an aggregate of identically prepared systems upon which measurements are performed or an alternating temporal sequence of preparations and measurements upon a single system, or a combination of both) and the collectives of measurement results obtained by real measurement operations upon its member systems. Moreover, assume that nothing is known about Newtonian properties or states of single systems. From this point of view, the sole purpose of classical statistical mechanics is to regularize within a causal framework purely probabilistic information about measurement results; hence the theory is a realization of the above paradigm and may be summarized as follows:

P1C: Functions $A(q,p)$, $B(q,p)$, $\cdots$ whose domain is phase space correspond to observables a, b, $\cdots$ [phase space has for coordinates the position (q), and momentum (p) components of the system of interest.]

P2C: For every ensemble of identically prepared systems there exists a mean value functional $m_t[A(q,p)]$. Every m_t may be expressed in terms of a corresponding function $\rho_t(q,p)$ (the density-of-phase) as follows:

$$m_t(A) = \int dq \int dp\, \rho_t(q,p) A(q,p).$$

(It can be shown that $\rho \geqq 0$, $\rho \not\equiv 0$, and $\int dq \int dp\, \rho = 1$.)

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[7] J. von Neumann, *Mathematical Foundations of Quantum Mechanics,* Eng. transl. by R. T. Beyer (Princeton University Press, Princeton, N.J., 1955), pp. 295–313.

P3C: For every kind of physical system, there exists a function $H(q,p)$ (the Hamiltonian) which determines the causal evolution of m_t via the following law (Liouville's equation):

$$\frac{\partial \rho_t}{\partial t} = \sum \left(\frac{\partial H}{\partial q}\frac{\partial \rho}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial \rho}{\partial q} \right) \equiv \{H, \rho\},$$

where Σ denotes summation over all conjugate pairs (q, p).

The pure and mixed ensembles of this theory are easily identified by combining P2C with the general definitions of pure and mixed. In terms of the density-of-phase function, an ensemble characterized by $\rho(q, p)$ is pure if there do not exist distinct (nonzero) functions $\rho^{(1)}(q, p)$, $\rho^{(2)}(q, p)$ such that

$$\rho(q,p) = w_1 \rho^{(1)}(q,p) + w_2 \rho^{(2)}(q,p), \quad w_1, w_2 > 0.$$

Now, it is obvious from analytic geometry considerations that any function $\rho(q, p)$ whose support (i.e., that part of the domain for which the function is nonzero) may be divided into two regions R_1 and R_2 can indeed be written as a linear combination of distinct functions $\rho^{(1)}$, $\rho^{(2)}$ by letting $\rho^{(1)}(\rho^{(2)})$ be proportional to ρ for points (q, p) in $R_1(R_2)$ and zero for points in $R_2(R_1)$. Thus the support of a *pure* $\rho(q, p)$ must be indivisible, i.e., a single point of phase space, say (q_o, p_o). The generalized function which satisfies this requirement plus the normalization condition $\int dq \int dp\, \rho(q, p) = 1$ is just the Dirac delta:

$$\rho(q,p) = \delta(q - q_o)\delta(p - p_o).$$

This is therefore the general form of ρ for the pure ensembles of classical statistical mechanics; particularly to be noted is the correspondence between these pure ensembles and the points (q_o, p_o) of phase space, a correspondence which will play a central role in subsequent development of a state concept applicable to a single system at a single time. Finally, the mixed ensembles of classical statistical mechanics are simply those represented by density functions not in the above pure form.

II. AUGMENTATION OF THE MODEL THEORY TO INCLUDE A STATE CONCEPT

The minimal axiomatic structure (P1, P2, P3) outlined above for an intrinsically probabilistic physical theory does not mention the concept of state. The omission was deliberate, for the usefulness of the state concept in physics lies in its reference to *individual* systems and its participation in the scheme of general causality (initial state of a system implies final state of that system via physical laws). Within the purely statistical model alone, such states play no role. There is, however, the possibility that they might be added somehow to that paradigm, provided the supplementation yields a consistent logical structure. It is demonstrated below that the concept of ensemble homogeneity provides both a suitable means for generating such a supplemental postulate and a touchstone to check for ambiguities or inconsistencies within a theory thus modified.

To motivate this general supplementation procedure, let us first reconsider classical statistical mechanics in the special form given above, where the axiomatic structure (P1, P2, P3) does not involve *a priori* any state concept for single systems. It is, however, well known that such a concept can indeed be consistently appended to this structure. In fact, the usual theoretical development of statistical mechanics begins with the classical states (q_o, p_o) and the Newtonian laws which govern their causal evolution; $\rho(q, p)$ is then defined so that $\int_{q1}^{q2} dq \int_{p1}^{p2} dp\, \rho(q, p)$ equals that fraction of an ensemble whose systems have classical states (q, p) in the phase space region $\{(q,p)|q\epsilon(q_1, q_2), p\epsilon(p_1, p_2)\}$, and Liouville's equation (P3C) is derived by applying classical mechanics to each member system of the ensemble.

In the present context, we are interested in the opposite procedure, viz., extracting from an originally statistical theory a consistent state concept for single systems. Thus in statistical mechanics the $\rho(q, p)$ is to be thought of initially only as characterizing an ensemble *in toto* and not as being formed by counting numbers of elements "really in" various classical states (q, p). The latter interpretation is what we want to "discover" from an analysis of measurement statistics associated with the ensembles. The fundamental question to be posed, then, is the following: Can an ensemble, described by $\rho(q, p)$, be construed unambiguously as consisting of elements to which some meaningful state specification may be assigned *individually*?

For classical statistical mechanics, the affirmative reply comes about as follows. Consider first the *pure* ensemble. Since, by definition, every division, or selection, produces subensembles statistically identical to each other and to the original, the pure ensemble provides the natural starting point for any additional postulate about *individual* states. Indeed, the homogeneity under division exhibited by the pure ensemble $\rho(q, p) = \delta(q - q_o)\delta(p - p_o)$ strongly suggests a supplemental postulate which assigns to *each* member system of such an ensemble a *state* (q_o, p_o), which may be thought of as possessed by the individual system. In classical statistical mechanics, the fact that observables are represented by *functions* $A(q, p)$ strengthens this suggestion; for thoughtful analysis clearly indicates that if *for every* $A(q, p)$, $m(A) = A(q_o, p_o)$ [as is the case when $\rho = \delta(q - q_o)\delta(p - p_o)$], the only reasonable definition for the possessed state of an individual system is the traditional phase space point (q_o, p_o) (or an equivalent specification in terms of the associated values of other phase functions).

This observation is, however, not yet sufficient to warrant inclusion of the proposed supplemental postulate; mixed ensembles have yet to be considered. Since *mixed* ensembles are characterized by reducibility to distinct subensembles and *pure* ensembles permit the unambiguous assignment of individual states, the fundamental question posed earlier may be replaced by another: Is the reduction of a mixed ensemble to a set of pure subensembles *unique*? If so, any ensemble may be consistently interpreted as a collection of systems *individually* described by definite states; if not, the proposed assignment of states to single systems is ambiguous and therefore physically meaningless.

In classical statistical mechanics, this reduction to pure subensembles is represented mathematically by expressing a general $\rho(q, p)$ as a "sum" of $\delta(q - q_o)\delta(p - p_o)$:

$$\rho(q, p) = \int dq_o \int dp_o \, c(q_o, p_o)\delta(q - q_o)\delta(p - p_o).$$

Necessarily, $c(q_o, p_o) = \rho(q_o, p_o)$, a *unique* solution. Therefore, every member of a classical ensemble may be unambiguously assigned a classical state (q_o, p_o) at any given time.

As noted above, the present discussion of statistical mechanics has proceeded retrograde to the traditional development. The ensemble and measurement statistics have been regarded as primitive, while the notion of state has been sought through an analysis of the over-all theoretical structure of the ensemble, in contrast to the standard procedure of forming ensembles from systems in mechanical states defined from the beginning. What has been done might be described rather crudely as the extraction of classical mechanics from statistical mechanics, instead of *vice versa.* More accurately, a study of the structure of classical statistics has shown that the state concept of classical mechanics may be "derived" by supplementing statistical mechanics with an additional state specification postulate, as opposed to the historic procedure of constructing statistical mechanics as the union of classical mechanics and statistics. To justify fully the identification of phase space points (q_o, p_o) as *states,* it is of course also necessary to prove that pure ensembles evolve into pure ensembles through Liouville's equation. This proof is not difficult; one simply demonstrates, by direct substitution, that $\rho(t) = \delta[q - Q(t)]\,\delta[p - P(t)]$ satisfies the Liouville equation, where $Q(t)$, $P(t)$ are the values of the canonical variables which evolve from their initial counterparts $Q(o) = q_o$, $P(o) = p_o$ via Hamilton's equations. To summarize, the concept *state* (of a single system at a single time) may be unambiguously attached to statistical mechanics as a fourth postulate:

P4C: Every system is always in a *state* represented by some point (q_o, p_o) of phase space.

The preceding discussion of classical statistical physics was given to illustrate the possibility of and the method for extracting the individual state concept from a theory initially concerned only with ensembles. As explained at the outset, the motivation for such an analysis comes from quantum theory, a discipline for which any sensible discussion of the state concept *must* proceed from the root notion of statistical ensemble owing to the intrinsic probabilistic nature of the theory. Thus the proper approach to the meaning of states in quantum theory is along the lines of the foregoing "reverse" development of statistical mechanics.

To study critically the nature of quantum states, we must therefore ascertain whether or not quantum theory, like statistical mechanics, is a realization of the general model given earlier (P1, P2, P3) augmented by a fourth postulate of the following form:

P4: Every system is always in a *state* represented by an element belonging to a set of mathematical objects which correspond to the pure ensembles of the theory.

The logical admissibility of P4 to any given theory always depends on the exact content of P1–P3. In particular, the state concept will be unambiguous if and only if the resolution of a general mixed ensemble into pure subensembles is *unique*. Moreover, if the state identification is to be physically meaningful, initially pure ensembles must retain their homogeneity under causal evolution.

III. IS QUANTUM THEORY A REALIZATION OF THE AUGMENTED PARADIGM?

At first glance, the augmented model seems tailor made for quantum theory; this should be no surprise, since it was obviously inspired by the standard axiomatic pattern of quantum-mechanics texts. However, to avoid jumping to conclusions, a stepwise analysis of the strictness of analogy between quantum theory and the general paradigm is in order. Accordingly, the postulates of quantum theory are scrutinized one by one and with an alertness to their consistency and interdependence.

P1Q: The linear Hermitian operators A, B, $\cdots$ on Hilbert space which have complete orthonormal sets of eigenvectors correspond to physical observables a, b, $\cdots$. If operator A corresponds to observable a, then the operator $F(A)$ corresponds to observable $F(a)$, where F is a function.

P2Q: For every ensemble of identically prepared systems there exists a real linear mean value functional $m_t(A)$ defined on the Hermitian operators.

Superficially, P1Q and P2Q seem almost too vague to have significant consequences. The vagueness is an illusion. In fact, P1Q and P2Q *imply* many well known features of quantum theory. In particular, it follows that every $m_t(A)$ may be expressed in terms of a corresponding operator ρ_t (the density operator) as follows: $m_t(A) = \mathrm{Tr}(\rho_t A)$. (It can be shown that ρ is Hermitian, positive semidefinite, and that $\mathrm{Tr}\rho = 1$.) Moreover, P1Q and P2Q imply that the probability for an a measurement to yield a result other than an A eigenvalue is zero. For a full discussion of this and other theorems, see Ref. 6.

There are several ways to express the dynamical postulate of quantum theory; perhaps the following version is best in the present context since we are specifically interested in contrasting quantum theory with classical statistical mechanics.

PQ3: For every kind of physical system, there exists an Hermitian operator H (the Hamiltonian) which determines the causal evolution of m_t via the following law:

$$i\hbar\,(\partial\rho_t/\partial t) = H\rho - \rho H \equiv [H, \rho].$$

P1Q, P2Q, and P3Q make up the axiomatic core of quantum theory; all the statistical results of quantum physics are derivable within this framework. Thus while a rather Procrustean reversal of foundations and consequences was required to force classical statistical mechanics into our model, quantum theory fits naturally. What seemed to be a "reverse" development in the classical case is the only sensible one in quantum physics, for there is no *nonprobabilistic* "mechanics" applicable to individual quantum systems. Since probabilities are present in quantum mechanics from the very outset, statistical ensembles are the physical objects of study from the very beginning.

Nevertheless, it is conceivable that some abstract object related to the pure ensemble might be theoretically attached to every quantum system as its state by an analysis parallel to that which led to P4C in the classical case. In short, quantum theory might be a realization of the augmented model (P1, P2, P3, and P4). To investigate this possibility, the pure ensembles of the theory must be found. In terms of the density operator ρ, an ensemble characterized by ρ is pure if there do not exist distinct (nonzero) positive semidefinite Hermitian operators $\rho^{(1)}$, $\rho^{(2)}$ such that

$$\rho = w_1\rho^{(1)} + w_2\rho^{(2)},\ w_1, w_2 > 0$$

The desired identification of the pure ensembles of quantum theory is accomplished by the following theorem due to von Neumann:

(H) ρ is pure if and only if $\rho = P_\psi$, where P_ψ is a projection operator onto the Hilbert vector ψ.

The proof, seldom repeated and perhaps unfamiliar to the reader, may be found in von Neumann's treatise.[8]

The theorem (H) establishes a correspondence between the pure ensembles of quantum theory and the points (vectors) of Hilbert space, a correspondence strikingly reminiscent of that between classical pure ensembles and the points of phase space. Thus once again it seems natural to take advantage of the full homogeneity of the pure ensemble—the defining characteristic that every subensemble is indistinguishable from the original by any measurement statistics—and assign a state to each individual system. In the present case, the vector ψ is clearly the appropriate representative for such a state. The stage is thus set for the seemingly harmless and perhaps intuitively useful jargon which makes statements of this type: "Suppose an electron is in the state ψ." Formally speaking, it seems reasonable to supplement P1Q, P2Q, P3Q by a P4Q which would assert that every quantum system is always in some state.

It would, however, be premature at this point to make such an addition to the quantal framework; first we must determine whether or not the proposed state specification scheme makes sense, i.e., whether or not it is ambiguous. As explained earlier, unambiguous individual state specification via the pure ensemble is possible only if the resolution of a general ensemble into *pure* subensembles is *unique*. Just as in classical statistical mechanics, this is a mathematical question strongly dependent on the precise content of P1, P2, P3. Since the pure form of the density operator has been identified as the projection P_ψ, what is in question is the uniqueness of the sum,

$$\rho = \sum_k w_k P_{\psi k}.$$

The surprising answer, which quantum theorists must face with all its ramifications, is the *negative* one. A general quantum ensemble can be subdivided in an infinite variety of ways into *pure* subensembles. As a result, the analogy between classical and quantal statistics breaks down. Quantum physics is *not* a realization of the foregoing augmented paradigm. There can be no "P4Q." Especially noteworthy is the consequence that it is generally improper to assign quantum-state vectors to individual systems. To do so ultimately leads only to paradoxes, as is illustrated below.

If all theoretical considerations in quantum theory could be carried out using only pure ensembles, the rather natural assignment of state vectors to single systems would be quite inconsequential. Thus, for instance, no bewildering paradoxes arise in the traditional applications of quantum mechanics wherein individual state specification is a commonplace notion. However, mixtures cannot always be circumvented; in particular, even if a set of given systems constitute a pure ensemble, the ensemble formed from specified *subsystems* of these is in general mixed. Hence the most general quantum ensembles—the mixtures—cannot be ignored.

To demonstrate that the assignment of state vectors to single quantum systems is essentially ambiguous and therefore improper, it suffices to give a simple illustrative counterexample.

Consider an ensemble of "spins," i.e., the associated Hilbert space is the familiar two-dimensional spinor space. Let α, β, δ, γ denote

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[8] Ref. 7, pp. 321–323.

eigenvectors of Pauli spin operators as follows:

$$\sigma_z\alpha = \alpha,\ \sigma_z\beta = -\beta,\ \sigma_x\delta = \delta,\ \sigma_x\gamma = -\gamma.$$

Let the statistical operator describing the ensemble at some given time be $\rho^{(1)} = \tfrac{3}{4} P_\alpha + \tfrac{1}{4} P_\beta$. It is obvious from the structure of $\rho^{(1)}$ that a proper selection will yield two subensembles divided as follows: one consisting of ¾ of the original systems and characterized by α, another made up of the remaining ¼ and characterized by β. One is inclined to describe this character of the ensemble by the statement that ¾ of the original systems are "in state α," ¼ "in state β." It is now shown that such an extrapolation from ensemble to single system leads to a paradox.

Consider a second ensemble of "spins" characterized by the statistical operator $\rho^{(2)} = \tfrac{3}{8} P_\delta + \tfrac{5}{8} P_\eta$, where $\eta = [1/(5)^{1/2}]\ (\delta + 2\gamma)$. This ensemble may be partitioned by selection into two pure subensembles: one made up of ⅜ of the original systems and characterized by δ, the second consisting of ⅝ of the initial ensemble and characterized by η. Following the natural procedure for state specification, the ensemble might then be described as a collection of systems ⅜ of which are "in the state δ," ⅝ "in the state η."

It is useful at this stage to exhibit the matrix representative of the operator $\rho^{(2)}$ in the $\{\alpha, \beta\}$-representation. The matrix elements are easily calculated: using $\delta = (1/\sqrt{2})\ (\alpha + \beta)$, $\gamma = (1/\sqrt{2}\ (\alpha - \beta)$, it follows that

$$(P_\eta) = \begin{bmatrix} <\alpha, P_\eta\alpha> & <\alpha, P_\eta\beta> \\ <\beta, P_\eta\alpha> & <\beta, P_\eta\beta> \end{bmatrix}$$

$$= \begin{bmatrix} 9/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix},\ (P_\delta) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix},$$

and

$$(\rho^{(2)}) = \tfrac{5}{8} \begin{bmatrix} 9/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix} + \tfrac{3}{8} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 3/4 & 0 \\ 0 & 1/4 \end{bmatrix}.$$

Hence,

$$(\rho^{(2)}) = \tfrac{3}{4} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \tfrac{1}{4} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

or

$$\rho^{(2)} = \tfrac{3}{4} P_\alpha + \tfrac{1}{4} P_\beta.$$

Comparison now shows that $\rho^{(1)} = \rho^{(2)}$; i.e., the statistical operators characterizing the two spin ensembles considered are equal. Since the statistical operator completely describes all measurement results for an ensemble, the equality of $\rho^{(1)}$and $\rho^{(2)}$ implies the *physical identity* of the ensembles they represent. Such an occurrence is by no means exceptional; this example is *not* a mathematical freak, or, as physicists sometimes say, a "pathological" case. Rather it illustrates a typical property[9] of quantum-mechanical ensembles, viz., that they are generally resolvable into *pure* subensembles in many ways, a property which was given above as the fundamental reason why quantum theory is not a realization of the augmented paradigm.

The basis for that conclusion may perhaps be clarified by further scrutinizing the present example: $\rho = \tfrac{3}{4} P_\alpha + \tfrac{1}{4} P_\beta = \tfrac{3}{8} P_\delta + \tfrac{5}{8} P_\eta$. From the structure of these resolutions of ρ into pure subensembles, it follows that there exist member systems which may belong either to a P_α ensemble or a P_δ ensemble. The following picturization of the ensembles involved is helpful in understanding this. Let each member system be denoted by a box so that the ensemble may be pictured as a row of boxes.

The resolution $\rho = \tfrac{3}{4} P_\alpha + \tfrac{1}{4} P_\beta$ means that the boxes may be rearranged and partitioned into two rows, one containing ¾ of the original boxes and physically describable by P_α, the second containing the remaining ¼ and physically describable by P_β.

Rearrangement must not be confused with physical interaction processes; what is considered here is the theoretical and conceptual structure of the ρ ensemble.

Similarly, the resolution $\rho = \tfrac{3}{8} P_\delta + \tfrac{5}{8} P_\eta$ means that the boxes may also be re-arranged and partitioned into two different rows, one containing ⅜ of the original boxes and physically describable by P_δ, the second containing the remaining ⅝ and physically describable by P_η.

A comparative examination of the two arrangements makes intuitively clear the above assertion that there are systems in the ρ ensemble which, without physical change, may belong to

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[9] E. Schrödinger, Proc. Cambridge Phil. Soc. 32, 446 (1936).

either a P_α ensemble or a P_δ ensemble. Now suppose for a moment the language of individual states is applied to these ensembles. Immediately a quantum monster is born: a *single* system concurrently "in" *two* states α and δ. To be explicit, the "state" of the system would have to be simultaneously an eigenvector of σ_z and σ_x, which is simply a mathematical impossibility. Thus the concept of individual quantum state is fraught with ambiguity and should therefore be avoided in serious philosophic inquiries concerning the nature of quantum theory.

IV. INDIVISIBILITY OF QUANTUM SYSTEMS

> "A new epoch in physical science was inaugurated . . . by Planck's discovery of the *elementary quantum of action,* which revealed a feature of *wholeness* inherent in atomic processes, . . ."[10]
>
> Niels Bohr

These words of the pioneer quantum philosopher from Copenhagen concisely express, by his own italics, the germinal idea from which the famous thesis of complementarity was born. Intrinsic *wholeness*—the *indivisibility* of quantum systems—is the essential ingredient of Bohr's philosophy. For almost 35 years he repeated the elements of his doctrine, at least part of which has come to be called the "orthodox" interpretation of quantum theory. However, Bohr was always content to philosophize in an almost wholly qualitative vein; even his illustrations from physics itself largely avoided the mathematics of quantum theory. Thus, in particular, the notion that quantum systems exhibit a peculiar indivisibility was always pleaded by deft application of uncertainty relations to primitive *gedankenexperiments.* Moreover, Bohr's depiction of quantum interaction depended strongly on an intuitive understanding of the behavior of "quanta" from the semiclassical perspective of what is often called the "Old Quantum Theory." It seems desirable therefore to provide the physical aspect of this indivisibility with a mathematical meaning in abstract quantum dynamics. First, however, in order to establish the origin of Bohr's idea of "wholeness," we review briefly some relevant points on the philosophic side of the indivisibility concept.

As we have seen, the laws of quantum theory do not and cannot refer to anything like classical states; quantum physics is in fact characterized by an extreme nonpicturability. Put simply, quantal laws govern the *statistics of measurement results,* and that is all. Obviously, such a theory is not immediately reconcilable with classic aspirations of natural philosophers for an *exhaustive* understanding of nature with "Cartesian clarity." Prominent among the dissenters was Einstein, who regarded the irreducible probabilities of quantum theory as an intolerable weakness, a glaring sign of incompleteness. Bohr combated this feeling for decades, often in specific encounters[11] with Einstein; again and again he argued that quantum theory, understood in terms of his complementarist philosophy, is indeed exhaustive.

Bohr's elaborations of that claim typically depart from the simple philosophic observation that science is concerned with intersubjective data. Thus, however bizarre the laws of microphysics may seem, experimental contrivances must themselves be described ultimately in some *communicable* manner. In actual practice, classical physics provides this language in which laboratory information is phrased in objective, unambiguous, communicable facts. It is a moot point whether this use of classical constructs is necessary or conventional; Bohr seemed to favor the first alternative. In any case, the requirement of intersubjectivity is in itself not peculiar to quantum theory.

What Bohr saw as an essentially quantum feature was rather "the introduction of a *fundamental distinction between the measuring apparatus and the objects under investigation,*"[12] a property which he regarded as a consequence of describing the apparatus in a language applicable to ordinary perceptions. As for the objects of study, quantum theory, the language suited to their description, defies visualization, as remarked earlier. Hence, says Bohr, the interaction between apparatus and object in quantum

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[10] N. Bohr, *Essays 1958-1962 on Atomic Physics and Human Knowledge* (Interscience Publishers, Inc., New York, 1963), p. 2.

[11] N. Bohr, *Atomic Physics and Human Knowledge* (John Wiley & Sons, Inc., New York, 1958), p. 32.

[12] Ref. 10, p. 3.

physics is an "inseparable part of the phenomenon. Accordingly, the unambiguous account of proper quantum phenomena must, in principle, include a description of all relevant features of the experimental arrangement."[13] No longer does physical theory either permit the neglect of or offer the means to compensate for interactions with the objects; yet all knowledge of quantum objects is obtained through interactions. From this concept of indivisibility, Bohr created the principle of complementarity, according to which the totality of results of different kinds of measurements on a quantum object *exhausts* all conceivable knowledge about such an object, even though these results cannot be combined to form a consistent *picture* of that object. Further elucidation of this principle would take us too far afield, since our present subject is not complementarity itself but the related conceptual indivisibility of interacting quantum systems.

The foregoing synopsis of Bohr's idea of the "wholeness" of quantum phenomena was effectively a paraphrasing of his views with no intentional distortion. Surveying his argument, one wonders whether Bohr has outlined a description or a derivation. Indeed a first impression might suggest the latter, as though the general requirement of communicability of data in conjunction with the quantal property of nonpicturability, i.e., the failure of classical microphysics, could imply a conceptual indivisibility of interacting systems. If such an implication were truly intended, then those critics of the Copenhagen interpretation who think of Bohr and Bishop Berkeley as two of a kind are correct, for such a "derivation" of indivisibility would indeed reflect the idealist dogma, *esse est percipi.* There is, however, a more favorable appraisal of Bohr's conception of wholeness in quantum theory. It is the understanding of his indivisibility notion as a philosophic description of the nature of quantum interactions, and not as a derivation of their nature from vague generalities.

To defend this proposal, let us consider the aforementioned *gedankenexperiments* which are typically used to enrich Copenhagen arguments by providing physical examples. Most famous of these thought experiments is Heisenberg's scheme for measuring electron position with a γ-ray microscope,[14] a token discussion of which is given in most elementary quantum mechanics textbooks. The electron position is to be observed microscopically under minimal illumination, i.e., by providing just one photon for the electron to scatter into the objective lens of the microscope. Physical optics requires short wavelength illumination, if a decent image of the electron is desired; but short wavelength means high frequency, hence a highly energetic photon. Upon collision, such a photon would of course transfer considerable momentum to the electron; thus it is said that the position measurement affects the electron momentum in an *unpredictable* and *uncontrollable* manner, limited only by the uncertainty relation $\Delta x \Delta p \sim (\lambda/\sin\theta)\ (h \sin\theta/\lambda) = h$, where Δx signifies the width of the image, Δp the unknowable momentum transfer, and θ is the angle with vertex at the electron which the scattered photon's path makes with the perpendicular from the electron to the lens.

Now, it is precisely this type of demonstration that Bohr takes as physical counterpart to his philosophical argument summarized above. The *unpredictable* effect of a position measurement on momentum suggests that the very concept of electron momentum should not even be contemplated for an electron interacting with a position-measuring device. Electron plus apparatus constitute a conceptual whole; to think of the electron independently is to divide the indivisible. So goes the complementarity argument.

Once again we ask whether this demonstration purports to derive or to describe. More quantitatively, does the microscope experiment *explain* on general grounds "why" $\Delta x \Delta p \sim h$ or is it only a picturesque *illustration* of the uncertainty relations for which *quantum theory is assumed at the outset*? The key to this question is the term *unpredictable,* which supposedly characterized the momentum transferred to the recoiling electron. Why is this quantity unpredictable, hence uncontrollable? Indeed Δp is calculable from the expression $\Delta p \sim h \sin\theta/\lambda$. The answer is simply

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[13] Ref. 10, p. 4.

[14] W. Heisenberg, *The Physical Principles of the Quantum Theory* (The University of Chicago Press, Chicago, Ill., 1930), p. 21.

that the quantum-theoretical approach has already been assumed; a *quantal* analysis of the pertinent collision problem could not predict θ. Therefore a logical textbook of quantum mechanics would not place the Copenhagen *gedankenexperiments* in their traditional first chapter home, but include them rather under applications of the theory (or perhaps in an appendix on the history of quantum mechanics.) Fock[15] is correct in pointing out that the mysterious term *uncontrollable* which is common in Copenhagen writings means nothing except that classical physics is inapplicable.

Thus it seems to us that what is really conveyed by Bohr's conception of indivisibility is a property of interaction in quantum theory, a property which can be clarified by proper identification of certain more abstract mathematical features of the theory and thereby comprehended in a manner more precise than crude inductions from *gedankenexperiments.* The following sections undertake such a program. First, classical statistical dynamics of interacting systems is formulated in a manner suitable for comparison with its quantum analog, which is developed subsequently. Finally, from the contrast between the classical and quantal cases, there emerges a mathematical meaning for quantum indivisibility.

V. CAUSAL EVOLUTION OF INTERACTING SYSTEMS

A. Classical Statistical Mechanics

The basic structure of classical statistical mechanics was reviewed in Secs. I and II. There the delta-function density of phase, $\rho = \delta(q - q_o)\delta(p - p_o)$, was identified as describing the *pure* ensemble of that theory. In the following analysis of interaction, we consider ensembles of composite systems, each consisting of two interacting subsystems. The dynamical evolution from two types of initial conditions is examined: (1) both systems initially in *pure* ensembles, (2) one system initially *pure,* the other *mixed.* The fundamental problem here posed is to determine the character (pure or mixed) of the ensembles to which the two systems belong after interacting for a time. Mathematically, *interaction* means that the Hamiltonian function $H(q_1, p_1, q_2, p_2)$ is not additive, i.e., there do *not* exist $V_1(q_1, p_1)$, $V_2(q_2, p_2)$ such that $H = V_1 + V_2$.

1. *Ensembles of Systems* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *Both Initially Pure:*

$$\mathbf{S}_1{:}\rho_1(q_1, p_1; t = 0) = \delta(q_1 - q_{10})\delta(p_1 - p_{10}).$$
$$\mathbf{S}_2{:}\rho_2(q_2, p_2; t = 0) = \delta(q_2 - q_{20})\delta(p_2 - p_{20}).$$
$$\mathbf{S}_1 + \mathbf{S}_2{:}\rho(t = 0) = \rho_1(t = 0)\rho_2(t = 0).$$

It will be intuitively obvious to anyone familiar with statistical mechanics that, under the transformation generated by the Liouville equation,

$$\rho(t = 0) \to \rho(t) = \delta[q_1 - Q_1(t)] \, \delta[p_1 - P_1(t)] \, \delta[q_2 - Q_2(t)] \, \delta[p_2 - P_2(t)],$$

where $Q_1(t)$, $P_1(t)$, $Q_2(t)$, $P_2(t)$ are the values of the canonical variables which evolve from their initial counterparts via Hamilton's equations. [This may be proved by direct substitution of $\rho(t)$ into the Liouville equation.]

Density functions ρ_1 and ρ_2 for the $\mathbf{S}_1$- and $\mathbf{S}_2$-ensembles are easily determined by integration:

$$\rho_1(q_1, p_1; t) = \delta[q_1 - Q_1(t)]\delta[p_1 - P_1(t)],$$
$$\rho_2(q_2, p_2; t) = \delta[q_2 - Q_2(t)]\delta[p_2 - P_2(t)].$$

Both represent pure ensembles; therefore two initially pure ensembles remain pure regardless of mutual interaction, as expected.

2. $\mathbf{S}_1$ *Ensemble Initially Pure,* $\mathbf{S}_2$ *Ensemble Initially Mixed*

$$\mathbf{S}_1{:}\rho_1(q_1, p_1; t = 0) = \delta(q_1 - q_{10})\delta(p_1 - p_{10})$$
$$\mathbf{S}_2{:}\rho_2(q_2, p_2; t = 0) = \int w(a, b)\delta(q_2 - a)\delta(p_2 - b)dadb,$$

where $w(a, b) = \rho_2(a, b; t = 0)$ is the initial probability density that an $\mathbf{S}_2$ is in classical state $(q_2, p_2) = (a, b)$,

$$\mathbf{S}_1 + \mathbf{S}_2{:}\rho(t = 0) = \delta(q_1 - q_{10})\delta(p_1 - p_{10}) \int dadb \, w(a, b)\delta(q_2 - a)\delta(p_2 - b)$$
$$= \int dadb \, w(a, b)\delta(q_1 - q_{10}) \, \delta(p_1 - p_{10})\delta(q_2 - a)\delta(p_2 - b),$$

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[15] V. Fock, Czech. J. Phys. 7, 642 (1957).

which is just a superposition of pure subensembles.

Using the linearity of Liouville's equation, we immediately obtain $\rho(t)$ for the present case by superposition. From case (1) we have

$$\begin{aligned}&\delta(q_1-q_{10})\delta(p_1-p_{10})\delta(q_2-a)\delta(p_2-b)\\ &\to\delta(q_1-Q_1)\delta(p_1-P_1)\delta(q_2-Q_2)\delta(p_2-P_2).\end{aligned}$$

Now, each of the functions, Q_1, P_1, Q_2, P_2, depends in general on the initial parameters q_{10}, p_{10}, a, b. Therefore, none of the δ's can be removed from the integral over a, b which superposes these pure densities. Hence

$$\begin{aligned}\rho(t)=\int dadb\, w(a,b)\delta(q_1-Q_1)&\\ \delta(p_1-P_1)\delta(q_2-Q_2)\delta(p_2-P_2)&,\end{aligned}$$

which, as is evident from its structure, represents a mixture. Moreover, *both* S_1 and S_2 ensembles are now mixtures:

$$\begin{aligned}\rho_1(t)=\int dadb\, w(a,b)\delta(q_1-Q_1)&\\ \delta(p_1-P_1),\cdots&\end{aligned}$$

Of special interest, the *initially pure* S_1 ensembles has been converted to a mixture by interaction with the initially mixed S_2 ensemble. This property of interaction, to be discussed later, is called *classical entanglement.*

B. Quantum Mechanics

We next consider the quantum-theoretical treatment of an ensemble each member of which is a composite of two interacting systems, S_1 and S_2. Mathematically, Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ are associated with these systems and $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the total Hilbert space for $S_1 + S_2$. The statement that S_1 and S_2 are interacting here means that there do *not* exist operators $V_1 \otimes 1$ and $1 \otimes V_2$ such that H, the total Hamiltonian operator for $S_1 + S_2$ may be written as $H = V_1 \otimes 1 + 1 \otimes V_2$.

From P3Q, the basic dynamical law of quantum mechanics is $i\hbar\ (\partial\rho/\partial t) = [H, \rho]$. The temporal transformation induced by this differential equation may always be succinctly expressed by a unitary operator $T(t)$ such that $\rho(t) = T(t)\ \rho(o)\ T^\dagger(t)$. Formally, $T(t) = \exp\ (-\ (i/\hbar) \int Hdt)$ or when H is independent of t, $T(t) = \exp\ [-\ (it/\hbar)\ H]$. In terms of the evolution operator $T(t)$, the statement that S_1 and S_2 are interacting means that there do *not* exist $T_1 \otimes 1$, $1 \otimes T_2$ such that $T = T_1 \otimes T_2$.

For the present problem, ρ is of course defined on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. To obtain ρ_1, ρ_2, the density operators for subsystems S_1, S_2, a "partial" trace operation is required. For example, $\rho_1 = Tr_2\rho$, which simply indicates a sum over those matrix elements of ρ whose two indices corresponding to $\mathcal{H}_2$ are identical.

A detailed exposition of the theory of composite systems may be found in von Neumann's work.[16] There, in addition to a fuller discussion of "partial" tracing, the following theorems, the first two of which are used below, are proved:

(a) If ρ_1 and ρ_2 are both pure, $\rho = \rho_1 \otimes \rho_2$ uniquely.
(b) If either ρ_1 or ρ_2 is pure, $\rho = \rho_1 \otimes \rho_2$ uniquely.
(c) If ρ_1 and ρ_2 are both mixed, ρ is not uniquely determined.

Now, working in parallel with the preceding section we determine the character (pure or mixed) of ensembles of interacting quantum systems S_1 and S_2.

1. Ensembles of Systems S_1 and S_2 Both Initially Pure:

$$\begin{aligned}S_1{:}\ \rho_1(t=0)&=P_{\phi 0}\\ S_2{:}\ \rho_2(t=0)&=P_{\chi 0}\\ S_1+S_2{:}\ \rho(t=0)&=P_{\phi 0}\otimes P_{\chi 0}=P_{\phi 0\otimes\chi 0}.\end{aligned}$$

The evolution operator $T(t)$ transforms $\rho(t = 0)$ into

$$\begin{aligned}\rho(t)&=T(t)P_{\phi 0\times\chi 0}T^\dagger(t)\\ &=|T(\phi_0\otimes\chi_0)><T(\phi_0\otimes\chi_0)|=P_{T(\phi 0\otimes\chi 0)}.\end{aligned}$$

Thus, as in the classical case, the over-all homogeneity of the ensemble is not altered by causal evolution. The density operators for the S_1 and S_2 ensembles are now determined by the partial trace equations. For this calculation, let $\{\phi_n\}$, $\{\chi_m\}$ be complete orthonormal sets in $\mathcal{H}_1$, $\mathcal{H}_2$, respectively, and consider the expansion

$$T(\phi_0\otimes\chi_0)=\sum_{nm}C_{nm}\,\phi_n\otimes\chi_m\equiv\psi.$$

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[16] von Neumann, Ref. 7, pp. 422–437.

Now,

$$\begin{aligned}\rho_1(t) &= Tr_2\, P_\psi \\ &= \sum_k < \chi_k \Big| \sum_{nm} C_{nm}\, \phi_n \otimes \chi_m > \\ &\qquad < \sum_{ls} C_{sl}\, \phi_s \otimes \chi_l \Big| \chi_k > \\ &= \sum_{knmls} \delta_{lk}\, \delta_{km}\, C^*{}_{sl}\, C_{nm} |\phi_n><\phi_s| \\ &= \sum_{nks} C^*{}_{sk}\, C_{nk} |\phi_n><\phi_s|,\end{aligned}$$

a mixture (since this is not in general a projection operator).

If S_1 and S_2 did not interact, i.e.,

$$H = V_1 \otimes 1 + 1 \otimes V_2,$$

then

$$\begin{aligned}&\exp\,[-\,(it/\hbar)\, H]\, (\phi_0 \otimes \chi_0) \\ &\quad = \exp\,[-\,(it/\hbar)\, V_1]\, \phi_0 \exp\,[-\,(it/\hbar)\, V_2]\, \chi_0,\end{aligned}$$

since $\exp\,(A + B) = \exp A \exp B$ if $[A, B] = 0$. Hence C_{nm} would have the product form $C_n\, C_m$, and then

$$\begin{aligned}\rho_1(t) &= \sum_{nks} C^*{}_s\, C^*{}_k\, C_n\, C_k |\phi_n><\phi_s| \\ &= \Big| \sum_n C_n\, \phi_n >< \sum_s C_s\, \phi_s \Big| \\ &\qquad\qquad \Big[\sum_k |C_k|^2 \Big] = P_\phi,\end{aligned}$$

where

$$\phi = \sum_n C_n\, \phi_n = \exp\,[-\,(it/\hbar)\, V_1]\, \phi_0.$$

Similar results hold for $\rho_2(t)$.

To summarize: unlike the classical case, two initially pure quantum ensembles are generally converted to mixtures during interaction; however, in the absence of interaction, both remain homogeneous. In the following section, we refer to this peculiarly quantum mixing as *quantum entanglement.*

2. S_1 *Ensemble Initially Pure,* S_2 *Ensemble Initially Mixed*

$$S_1 : \rho_1(t = 0) = P_{\phi 0}$$

$$S_2 : \rho_2(\, t = 0) = \sum_k W_k\, P_{\eta k},\ \sum_k W_k = 1,$$

$$S_1 + S_2 : \rho(t = 0) = P_{\phi 0} \otimes \sum_k W_k\, P_{\eta k}.$$

As in the classical case (2), linearity of the dynamical law enables solution by superposition. Thus, from

$$T(t)\ (P_{\phi 0} \otimes P_{\eta k})T^\dagger(t) = \rho_k(t) = P_{\psi k},$$

where

$$\psi_k = \sum_{nm} C^{(k)}_{nm}\, \phi_n \otimes \chi_m,$$

it follows that

$$\rho(t = 0) \rightarrow \rho(t) = \sum_k W_k\, P_{\psi k};$$

i.e., ρ, initially a mixture, in general remains inhomogeneous.

Of great interest here is the effect of interaction on the initially pure S_1 ensemble. Taking the appropriate trace, we find, using the result from case (1),

$$\begin{aligned}\rho_1(t) &= \mathrm{Tr}_2\, \rho(t) = \sum_k W_k\, \mathrm{Tr}_2\, P_{\psi k} \\ &= \sum_k W_k \Big[\sum_{rsn} C^{(k)*}_{sr}\, C^{(k)}_{nr}\, |\, \phi_n >< \phi_s\, | \Big],\end{aligned}$$

which in general represents a mixture.

To clarify the nature of this latter transformation,

$$P_{\phi 0} \rightarrow \sum_k W_k \Big[\sum_{rsn} C^{(k)*}_{sr}\, C^{(k)}_{nr}\, |\, \phi_n >< \phi_s\, | \Big],$$

let us suppose quantum entanglement did not exist. Then case (1) would have yielded

$$P_{\phi 0} \otimes P_{\eta k} \rightarrow P_{\phi(\phi 0, \eta k)} \otimes P_{\eta(\phi 0, \eta k)};$$

superposing these, we obtain

$$\rho(t) = \sum_k W_k\, P_{\phi(\phi 0, \eta k)} \otimes P_{\eta(\phi 0, \eta k)}.$$

Thus

$$\begin{aligned}\rho_1(t) &= \sum_k W_k \sum_n < \chi_n | P_\phi \otimes P_\eta | \chi_n > \\ &= \sum_k W_k\, P_{\phi(\phi 0, \eta k)}\, \mathrm{Tr}_2\, P_{\eta k} = \sum_k W_k\, P_{\phi(\phi 0, \eta k)},\end{aligned}$$

which we recognize as analogous to the effect previously called classical entanglement.

Returning now to the correct result,

$$\rho_1(t) = \sum_k W_k \Big[\sum_{rsn} C^{(k)*}_{sr}\, C^{(k)}_{nr}\, |\, \phi_n >< \phi_s\, | \Big]$$

we identify the mixtures in parentheses as a consequence of quantum entanglement, while the further mixing by the sum over k corresponds to classical entanglement.

VI. CLASSICAL DIVISIBILITY AND QUANTUM INDIVISIBILITY FROM A DYNAMICAL VIEWPOINT

In a certain sense, the foregoing deductions (Sec. V-A) concerning classical interactions are paradoxical. *For any Hamiltonian whatsoever*, even one which includes interaction terms, the homogeneity of an ensemble of systems interacting with members of a second initially pure ensemble is preserved throughout the motion; nevertheless, by arranging an interaction between the members of an initially pure ensemble and those of a mixture, the initially homogeneous ensemble in general changes into an *inhomogeneous* ensemble after a time, a process we called classical entanglement. Therefore we confront a strange dilemma, for in classical entanglement mere juxtaposition apparently achieves the impossible, viz., the destruction of homogeneity, which no Hamiltonian, hence no force conceivable within this theory, can accomplish. To explain this seeming discrepancy, consider a simple problem in classical physics.

Let S_1 and S_2 each be a particle of mass m, interacting via a connecting spring of stiffness k and equilibrium length l. The Hamiltonian function is therefore $H = (1/2m)\ (p_1{}^2 + p_2{}^2) + (k/2)\ (q_2 - q_1 - l)^2$, where p_1, p_2 denote linear momenta and q_1, q_2 are position coordinates for S_1, S_2. From Hamilton's equations it is easy to obtain Newton's law:

$$m\ddot{q}_1 = k(q_2 - q_1 - l),$$

$$m\ddot{q}_2 = -k\ (q_2 - q_1 - l).$$

When the latter pair of coupled differential equations are solved subject to the initial conditions $q_1(o) = 0$, $\dot{q}_1(o) = \dot{q}_2(o) = 0$, $q_2(o) = q_{20}$ the result is $q_1(t) = \tfrac{1}{2}\ (q_{20} - l)\ [1 - \cos \sqrt{2}\, wt]$, $w = (k/m)^{1/2}$.

Now, a relevant question is whether there exists some force which, when acting on S_1 produces the same motion $q_1(t)$ as did the above interaction. Certainly. In fact the Hamiltonian

$$H_1 = p_1{}^2/2m + \tfrac{1}{2}\ (2k)\ [q_1 - (q_{20} - l)/2]^2$$

produces the desired motion. [Physically, S_1 could be at one end of a spring of stiffness $2k$ and equilibrium length $(q_{20} - l)/2$ which is rigidly mounted at its other end.]

Consider an initial S_2 ensemble mixed by being distributed over the values of q_{20}. By classical entanglement, an initially pure S_1 ensemble interacting with this mixed one will itself become inhomogeneous. The reason for this is now easily explained by the observation that the Hamiltonian H_1 *depends on* q_{20}. Thus the S_1 "ensemble," when considered by itself, is not an ordinary *ensemble* at all; for it does not consist of a collection of identical systems but rather of a distribution of *different kinds of systems*, characterized by *different* Hamiltonians $H_1(q_{20})$. (Physically, the various S_1 could be attached to springs with differing equilibrium lengths.) It is therefore not alarming that this "ensemble" behaves in a manner contradictory to the Liouville equation.

The paradox of classical entanglement was illusory; the "impossible" disruption of homogeneity in the S_1 ensemble had a theoretical explanation. Thus, the existence of classical entanglement is no reason to declare interacting systems S_1 and S_2 to be in any sense indivisible.

For quantum entanglement, the situation is quite different. Here no explanation parallel to that given for the classical case is possible. Quantal entanglement occurs under conditions of *maximal homogeneity* (S_1 and S_2 both initially pure); but it was the *inhomogeneity* of S_2 and its reflection in H_1 that made classical entanglement "divisible", hence explicable. In the quantum case there is no way to explain away the basic paradox of entanglement, viz., that interacting systems S_1 and S_2 each from initially pure ensembles develop temporally into members of mixed ensembles. Since there *does not exist* an evolution operator T such that $TP_{\phi_0}T^\dagger = \rho_t$ where ρ_t is not a projection operator, no physical environment conceivable within quantum theory has the same effect on the initially pure S_1 ensemble as quantum entanglement with the S_1 ensemble. Thus, in an explicit dynamical sense, interaction in quantum theory exhibits a remarkable property of "wholeness," to use Bohr's word.

If S_1 is an atomic system and S_2 the measuring device through which S_1 is studied via interaction, then it is in fact a quantum dynamical property of this interaction that it becomes impossible —even with quantum theory—to give an *independent* account of the temporal development of S_1. The composite system $S_1 + S_2$, quantally entangled, may therefore be regarded as dynamically "indivisible." With this mathematical interpretation, perhaps Bohr's concept of "wholeness," a fundamental attribute of quantum theory and a pillar of complementarity, obtains a more definitive meaning.

VII. SUMMARY: STATE CONCEPT IN QUANTUM THEORY

Ever since Born first provided quantum theory with its fundamental link to nature via statistics, controversy has raged over the extent to which this innovation modifies the basic classical structure of physical science. Since the only quantal constructs which participate in a causal law relate to nature solely through probabilistic-statistical rules of correspondence, it is obvious that any "state" concept in quantum theory must refer *empirically* to statistical ensembles instead of individual systems. By this we mean simply that the "states" of quantum theory are related to statistical collectives of measurement results emerging from measurements upon identically prepared systems. The only sense in which such "states" might be construed as referring empirically to a single system is in the case of an ensemble consisting of *one* system sequentially measured and reprepared; however, this is beside the point in the present inquiry, which has sought to ascertain whether or not one may consistently regard quantum "states" as belonging to physical systems in the classical manner wherein every system is thought of as *always being in some definite* (possibly unknown) *state.* The fact that quantum states refer empirically to ensembles does not preclude the *theoretical* possibility of restoring a causal nexus for individual states, as was demonstrated above for the case of "retrograde" classical statistical mechanics. However, that possibility was found to hinge upon (1) the question of uniqueness in the problem of resolution of general ensembles into *pure* subensembles, since the pure ensemble is the appropriate construct from which to develop an individual state specification for an initially statistical theory; and (2) the question of "conservation of homogeneity" for an initially pure ensemble under causal evolution, since any meaningful individual state concept should be applicable to a system at all times, i.e., it should be possible to follow the temporal development of the state via the causal law of the theory.

The preceding sections sought the meaning of quantum states within the minimal logical framework of quantum mechanics; accordingly, the analysis centered on the ensembles which give rise to the statistical collectives of measurement results actually studied in quantum theory. No quantum state concept was assumed *a priori;* instead we began only with the elementary assumption that there are systems upon which measurements of observables are performed according to established rules of correspondence, and that the statistics of the numerical results which emerge from such measurements upon an ensemble of identically prepared systems are governed by quantum theory, (P1Q, P2Q, P3Q). Against this background the logical standing of the usual theoretical quantum state concept was assayed using the above criteria (1) and (2).

The results of the analysis, with comparisons to the analogous classical situation, may be summarized as follows:

(1) Decomposition of the classical density-of-phase into representatives of pure subensembles is unique; thus single classical systems may be assigned states (q_o, p_o) unambiguously. On the other hand, resolution of quantum ensembles into pure subensembles is not unique; hence, as illustrated earlier, the assignment of a state vector ψ to a single system at a single time is an ambiguous procedure which can lead to theoretical paradoxes.

(2) A classical pure ensemble, even when interacting with another pure ensemble, remains at all times pure; it makes sense therefore to assign a temporal sequence of states $[q(t), p(t)]$ to any classical system. On the contrary, although a pure ensemble of closed quantum systems does conserve its homogeneity, a quantum pure ensemble interacting with another pure ensemble becomes entangled and is converted to a mixture; thus it is in general impossible to

assign a temporal sequence of state vectors $\psi(t)$ to a quantum system.

In short, quantum theory satisfies neither of the two criteria necessary to justify the supplementation of an initially statistical theory by a state concept in the classical sense. Although classical statistical mechanics admits of an unambiguous assignment of individual states, quantum theory fails to satisfy the necessary criteria. Hence the simplest, and most natural, conclusion is that the pure "state" vector ψ of quantum theory must not be interpreted—even theoretically—as referring to the physical state of a single system at a single time. This does not imply that the basic meaning of causality is lost in quantum physics. Although the classical ideal of determinism as applied to single physical events is not valid in quantum mechanics, the behavior of quantum ensembles is predictable in the sense that future measurement statistics are determined by past measurement statistics. Thus, strictly speaking, the state concept in quantum mechanics belongs to the ensemble instead of the system. Indeed *quantum systems should be regarded as never being in any physical state* (except in the aforementioned statistical sense where the "state" refers to a single system because the ensemble consists of one system sequentially measured and reprepared). To use a terminology sometimes used in statistical theories, quantum mechanics may be characterized as *a theory with macrostates* (*of ensembles*) *for which there are no underlying microstates* (*of systems*).

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Coordinate Transformations of W. Voigt and the Principle of Special Relativity

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The relationship existing between the Voigt transformation and the Lorentz transformation is discussed. Some properties of the Voigt transformation are considered, especially its conformal nature. The mechanical meaning of the Voigt transformation is treated, i.e., the moving oblate spheroid, and a moving clock. The addition of velocities theorem developed by application of the special Voigt transformation and the addition of velocities theorem developed by application of the Lorentz transformation are shown to be identical. Finally, the mechanical necessity that the special Voigt coordinate transformations be altered in order to construct the Lorentz electron model, and the electrodynamic consequences of the Voigt kinematics are discussed.

INTRODUCTION

RECENT discussion concerning the Voigt coordinate transformations, most notable of which is that by M. Strauss,[1] has stimulated some interest in their kinematical and electrodynamical consequences. Part of what Waldemar Voigt[2] did in 1887 was to consider the propagation of oscillating disturbances through an elastic uniform incompressible medium; and in so doing, derived a set of coordinate transformations which occupy a unique place in the history of science. These he applied to the study of the Doppler principle, in a manner not too unlike

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[1] M. Strauss, Nuovo Cimento 39, 658 (1965).

[2] W. Voigt, K. Ges. Wiss. Gött. 4, 41 (1887) LC AS 182. G 82; reprinted Physik Z. 16, 381 (1915).