A stochastic derivation of the Klein–Gordon equation

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Several years ago Nelson succeeded in deriving the nonrelativistic Schrödinger equation within a stochastic model which included Newton's second law as the fundamental dynamical rule. Unfortunately, the relativistic extension of Nelson's work is not so straightforward as might at first be supposed. This paper examines the difficulties inherent in such a relativization and proposes supplemental axioms which resolve those difficulties. A stochastic derivation of the Klein–Gordon equation is then presented.

1. INTRODUCTION

Quantum mechanics and the theory of stochastic processes are normally perceived as being rather separate branches of physics. It is possible, however, to establish a formal relationship between them by exploiting an obvious similarity, viz., that each embodies the concept of probability as an irreducible element in its axiomatic framework. The most widely cited effort along these lines was made by Nelson,[1] who succeeded in deriving the nonrelativistic Schrödinger equation as a theorem within a stochastic model based upon Newton's second law. Nelson[2] suggested later that it might be possible to extend his techniques to the relativistic case, but no extension of this kind has so far been published.

In the present paper we develop a method for obtaining such an extension, and compare our results with the work of other authors[3,4] who have attempted alternative stochastic derivations of relativistic wave mechanics.

2. RUDIMENTS OF STOCHASTIC MECHANICS

The stochastic particle of interest is regarded as a classical punctiform mass, occupying at every instant a single point in space and traveling, therefore, along a trajectory. The probabilistic element, which is essential to provide a link to quantum mechanics, is introduced by assuming that this trajectory is continually influenced by a hidden thermostat similar, for example, to those suggested by Bohm[5] and deBroglie[6] in connection with hidden-variables theories. While the exact properties of the thermostat-particle interaction are unknown, the fluctuations of the particle position resulting from this interaction are presumed to be describable as a Markoff process.

Accordingly, the position probability density $\rho(\mathbf{x}, t)$ must obey the Smoluchowski equation

$$\rho(\mathbf{x}, t+\Delta t) = \int P(\mathbf{x}-\Delta\mathbf{x}, t \mid \Delta\mathbf{x}, \Delta t)\rho(\mathbf{x}-\Delta\mathbf{x}, t)d^3(\Delta x), \tag{1}$$

where $P(\mathbf{x}-\Delta\mathbf{x}, t \mid \Delta\mathbf{x}, \Delta t)$ is the conditional probability density that a particle at position $\mathbf{x}-\Delta\mathbf{x}$ at time t will be displaced by $\Delta\mathbf{x}$ during the interval Δt, thus reaching position $\mathbf{x}$ at time t.

Similarly, let $F(\mathbf{x}+\Delta\mathbf{x}, t \mid \Delta\mathbf{x}, \Delta t)$ denote the probability density that a particle with position $\mathbf{x}+\Delta\mathbf{x}$ at time t has been displaced through $\Delta\mathbf{x}$ in the preceding interval Δt and thus would have been found at $\mathbf{x}$ at the earlier instant $t-\Delta t$. The analog of Smoluchowski's equation with F instead of P is therefore

$$\rho(\mathbf{x}, t-\Delta t) = \int F(\mathbf{x}+\Delta\mathbf{x}, t \mid \Delta\mathbf{x}, \Delta t)\rho(\mathbf{x}+\Delta\mathbf{x}, t)d^3\Delta x. \tag{2}$$

Since the motion of the particle is to be conceived as a stochastic process, it follows, strictly speaking, that $\mathbf{x}(t)$ is not differentiable. To meet this difficulty, Nelson suggested two possible alternatives for the time derivative of position, the mean forward derivative $D\mathbf{x}(t)$ and the mean backward derivative $D^*\mathbf{x}(t)$, defined respectively as

$$D\mathbf{x}(t) \equiv \lim_{\Delta t\to 0} \frac{\langle \mathbf{x}(t+\Delta t)-\mathbf{x}(t)\rangle}{\Delta t} \equiv \lim_{\Delta t\to 0}\int\left(\frac{\mathbf{x}(t+\Delta t)-\mathbf{x}(t)}{\Delta t}\right) P(\mathbf{x}, t \mid \Delta\mathbf{x}, \Delta t)d^3(\Delta x), \tag{3}$$

$$D^*\mathbf{x}(t) \equiv \lim_{\Delta t\to 0} \frac{\langle \mathbf{x}(t)-\mathbf{x}(t-\Delta t)\rangle^*}{\Delta t} \equiv \lim_{\Delta t\to 0}\int\left(\frac{\mathbf{x}(t)-\mathbf{x}(t-\Delta t)}{\Delta t}\right) F(\mathbf{x}, t \mid \Delta\mathbf{x}, \Delta t)d^3(\Delta x). \tag{4}$$

By hypothesis the motion $\mathbf{x}(t)$ is regarded as being separable into two parts, an ordinary functional part and a Wiener process $\omega(t)$, i.e.,

$$d\mathbf{x}(t) = \mathbf{b}[\mathbf{x}(t), t]\,dt + d\omega(t), \tag{5}$$

where

$$\langle d\mathbf{x}\rangle = \mathbf{b}\,dt, \tag{6}$$

$$\langle (d\mathbf{x})^2\rangle = \langle (d\omega)^2\rangle + O(dt^2), \tag{7}$$

or,

$$\langle (d\mathbf{x})^2\rangle \approx 2\nu\,dt, \tag{8}$$

where ν is, by definition, the diffusion constant.

The stochastic derivative of a function $f(\mathbf{x})$ is given by means of Ito's rule[7]:

$$Df(\mathbf{x}) \equiv \left(\frac{\partial}{\partial t} + \mathbf{b}\cdot\nabla + \nu\nabla^2\right) f(\mathbf{x}). \tag{9}$$

The value of ν to be selected in any particular realization of stochastic mechanics depends upon the nature of the thermostat. In order to match the predictions of nonrelativistic quantum mechanics, Nelson chose $\nu = 3\hbar/2m$.

For the backward case, there are relations analogous to Eqs. (5)—(8), viz.,

$$\langle d\mathbf{x}\rangle^* = \mathbf{b}^*\, dt, \tag{10}$$

$$\langle (\mathrm{d}x)^2\rangle^* \approx \nu\, dt = (3\hbar/2m)\, dt, \tag{11}$$

$$D^* f(\mathbf{x}) = \left(\frac{\partial}{\partial t} + \mathbf{b}\cdot\nabla - \nu\nabla^2\right) f(\mathbf{x}). \tag{12}$$

By using the mathematical properties of $\mathbf{x}(t)$ described in Eqs. (5)—(12), it is now possible to generate Taylor expansions of the integrands in the Smoluchowski equations (1) and (2) and thereby extract the normal Fokker—Planck equation

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\mathbf{b}) + \frac{\hbar}{2m}\nabla^2\rho \tag{13}$$

and its backward counterpart

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\mathbf{b}^*) - \frac{\hbar}{2m}\nabla^2\rho. \tag{14}$$

Adding (13) and (14) then yields the continuity equation

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\mathbf{v}), \tag{15}$$

where $\mathbf{v}$ denotes the average of $\mathbf{b}$ and $\mathbf{b}^*$. The quantities $\mathbf{b}$, $\mathbf{b}^*$, and $\mathbf{v}$ are called, respectively, forward, backward, and total drift velocity.

3. FORMULATION OF THE RELATIVISTIC CASE

To construct a specific quantal wave equation from the elements of the general stochastic mechanics just reviewed requires the adoption of some specific dynamical rule. Thus to obtain the Schrödinger equation, Nelson assumed the Newtonian rule $\mathbf{F} = m\mathbf{a}$, the acceleration $\mathbf{a}$ being defined as follows:

$$\mathbf{a}(t) \equiv \tfrac{1}{2}(DD^* + D^*D)\mathbf{x}(t) = \lim_{\Delta t\to 0}\frac{1}{2}\left(\frac{\langle \mathbf{b}(t) - \mathbf{b}(t-\Delta t)\rangle^*}{\Delta t} + \frac{\langle \mathbf{b}^*(t+\Delta t) - \mathbf{b}^*(t)\rangle}{\Delta t}\right). \tag{16}$$

Kracklauer[8] has criticized this definition of $\mathbf{a}$ on the ground that Nelson failed to provide an adequate physical rationale for it. However, careful scrutiny[9] of the operational meaning of an acceleration measurement does indeed produce the desired rationalization and rebut Kracklauer's objection to Nelson's theory.

By combining this stochastic version of Newton's second law with the continuity equation, Nelson derived two real equations which are equivalent to the complex Schrödinger equation.

At first glance one might conjecture that the relativistic wave equations of quantum mechanics should be similarly derivable simply by substituting for the $\mathbf{F} = m\mathbf{a}$ of Nelson's theory the analogous rule in classical relativistic mechanics. There are, however, certain unexpected difficulties.

First, Hakim[10] has shown that, if the limit $\Delta t\to 0$ is used to calculate conditional probability densities like P and F, the only value for the diffusion constant ν compatible with relativistic invariance is zero. To circumvent this difficulty, we shall discretize the time variable in the stochastic description, so that in the sequences of events which define trajectories, adjacent events have a nonzero minimum temporal separation τ. Obviously the value of this time period must be sufficiently small to ensure that, in any "practical" reference frame, there can be no conflict between the predictions of the stochastic theory and actually realizable macroscopic observations. A specific value for τ will be given later when we develop the model in detail.

A second difficulty in extending Nelson's work arises in the fact that the nonrelativistic formulation assumes that the particle could traverse even an infinite distance in a finite period of time, since both $P(\mathbf{x}, t\,|\,\Delta\mathbf{x}, t)d^3(\Delta x)$ and $F(\mathbf{x}, t\,|\,\Delta\mathbf{x}, \Delta t)d^3(\Delta x)$ are nonzero for the free particle no matter how large $|\Delta\mathbf{x}|$ becomes. To eliminate these spacelike trajectories from the theory, we propose to restructure the stochastic development around a postulate inspired by a curious feature of Dirac's relativistic theory of the electron.[11] According to the latter, the operator for the magnitude of the instantaneous velocity of a free electron possesses just one eigenvalue, the speed c of light in vacuo. Thus if Dirac's theory is correct, the only possible result of such a speed measurement on a free electron would indeed be c. Dirac himself rationalized this proposition through an ambiguous discussion of the Heisenberg uncertainty principle. However, we simply adopt it as an axiom of the stochastic model, i.e., we postulate that c is the instantaneous speed of the relativistic stochastic particle between interactions with the hidden thermostat. Of course the particle cannot travel too far at the velocity of light, for such behavior would surely be incompatible with known facts. In the Dirac theory, the particle executes the physically unexplained Zitterbewegung. In our stochastic model such an oscillation will also appear, but with an evident physical cause, viz., the interactions with the thermostat. That is, the particle travels for a short period of time at the speed of light, then interacts with the thermostat, instantaneously changing direction but not speed.

To summarize, our stochastic derivation of quantum dynamics is to be founded upon Nelson's postulates supplemented by two special axioms: (i) the discretization of time in the stochastic model, and (ii) the attribution of the speed of light to the stochastic particle between interactions with the thermostat.

Assumption (i) must now be made more explicit. We begin by recalling (8) and Nelson's choice of $\nu = 3\hbar/2m$, which leads in the nonrelativistic case to the relation

$$\lim_{\Delta t\to 0}\frac{1}{2}\frac{\langle (d\mathbf{x})^2\rangle}{\Delta t} = \frac{3\hbar}{2m}. \tag{17}$$

In order to avoid the difficulty mentioned by Hakim, (17) must be changed in the relativistic description to

$$\lim_{\Delta t\to\tau}\frac{1}{2}\frac{\langle (d\mathbf{x})^2\rangle}{\Delta t} = \frac{3\hbar}{2m} \tag{18}$$

or,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}\frac{(\Delta\mathbf{x}_i)^2}{\tau} = \frac{3\hbar}{2m}, \tag{19}$$

where $\Delta\mathbf{x}_i$ represents the ith sample displacement in an ensemble of n free excursions of the particle between interactions with the thermostat. To determine the value of the free travel time τ, we note that assumption (ii) implies

$$|\Delta \mathbf{x}_i|/\tau = c, \quad \forall \Delta \mathbf{x}_i \tag{20}$$

which, combined with (19), yields

$$\tfrac{1}{2}c^2\tau = 3\hbar/2m \tag{21}$$

or,

$$\tau = 3\hbar/mc^2. \tag{22}$$

Interestingly, this free flight time τ is exactly $3/\pi$ times the Zitterbewegung period of Dirac's electron theory; it differs by a factor of $3/2\pi$ from deBroglie's intrinsic vibration period for a quantum particle.[12] It is important to note that τ is a fixed quantity, not an average; thus in each interval τ, the particle travels exactly the free path length

$$\lambda \equiv c\tau = 3\hbar/mc. \tag{23}$$

Hence the behavior of the relativistic stochastic particle characterized by assumptions (i) and (ii) is in fact a physical realization of the mathematical random walk concept.

Some interesting speculation can be carried out by examining the free path length of various particles. For electrons, $\lambda \approx 10^{-10}$ cm; and for protons, $\lambda \approx 10^{-14}$ cm. If we now compare these lengths to the effective range of the strong nuclear force ($\sim 10^{-13}$ cm), we see that an electron could participate in the nuclear interaction only if its behavior violated our random walk model, whereas the proton could be influenced by the nuclear force without contradicting our assumptions. Thus the proton could change direction at free path endpoints in response to the nuclear interaction as well as to the hidden thermostat. It is of course an experimental fact that the proton, and not the electron, is affected by the strong nuclear interaction.

Finally, it is necessary to specify the Lorentz frame of reference relative to which the stochastic particle executes the proposed random walk, with characteristic values for period τ and length λ given by (22) and (23). We shall take this frame to be the drift rest frame defined precisely in the next section. Physically, the velocity of the drift rest frame relative to an observer is the velocity v the observer would assign to the stochastic particle on the basis of ordinary measurement, in contradistinction to the empirically unmeasurable instantaneous velocity (of light) actually possessed by the stochastic particle in each of its free excursions. To be assured that the assumed value for τ is small enough so that its existence would not contradict known facts, it suffices to note that even for an electron ($\tau \approx 10^{-21}$ sec) that had been accelerated to $v = 0.999c$, the time-dilated interval between interactions with the hidden thermostat would still be only

$$\Delta t = \frac{\tau}{(1 - v^2/c^2)^{1/2}} \approx 10^{-19} \text{ sec}. \tag{24}$$

4. DERIVATION OF RELATIVISTIC QUANTUM WAVE EQUATIONS

It will now be demonstrated that the proposed relativistic extension of stochastic mechanics leads to two real equations which are equivalent to the complex quantum equation for the spinless relativistic particle, the Klein–Gordon equation. Moreover, since each spin component of the Dirac equation satisfies the Klein–Gordon equation, this approach could be considered as explaining the motion of half-integral-spin particles when the spin can be neglected.

In Nelson's derivation of the Schrödinger equation from nonrelativistic stochastic mechanics, conditional probability distributions P and F had to give probability estimates over all space for the future and past locations of the particle. However, in the relativistic random walk particle, given that the particle is at $\mathbf{x}$ at time t, the only possible positions for it at time $t-\tau$ and $t+\tau$ are those on the sphere of radius λ surrounding $\mathbf{x}$. Thus we only need to define the probabilities $P_R(\Omega)\,d\Omega$ and $F_R(\Omega)\,d\Omega$ that the particle will go to or came from the area of unit solid angle Ω to $\Omega + d\Omega$. In general, P_R and F_R will be dependent on the past history of the particle but it will be assumed that we may describe the particle motion by a Markoff process so that we can write $P_R(\Omega) = P_R(\mathbf{x}, t\,|\,\hat{n}(\Omega), \tau)$ and $F_R(\Omega) = F_R(\mathbf{x}, t\,|\,\hat{n}(\Omega), \tau)$, where $\hat{n}(\Omega)$ is a unit vector and $P_R(\mathbf{x}, t\,|\,\hat{n}(\Omega), \tau)$ describes the probability density that the particle located at $\mathbf{x}$ at time t will be found at time $t+\tau$ in the position $\mathbf{x} + \Delta\mathbf{x} = \mathbf{x} + \lambda\hat{n}(\Omega)$. A similar definition holds for $F_R(\mathbf{x}, t\,|\,\hat{n}(\Omega), \tau)$. In analogy to the nonrelativistic case, we can define a relativistic forward drift 4-velocity $b_\mu = (b_0, b_i)$, where

$$b_i = b^i \equiv \frac{\lambda}{\tau}\int n_i(\Omega) P_R(\mathbf{x}, t\,|\,\hat{n}(\Omega), \tau)\, d\Omega$$

$$= c\langle n_i\rangle, \tag{25}$$

$$b_0 = -b^0 \equiv -c(\tau/\tau) = -c. \tag{26}$$

Similarly, we may introduce a backward drift 4-velocity $b_\mu^* = (b_0^*, b_i^*)$, where

$$b_i^* \equiv \frac{\lambda}{\tau}\int n_i F_R(\mathbf{x}, t\,|\,\hat{n}(\Omega), \tau)\, d\Omega$$

$$= c\langle n_i\rangle^*, \tag{27}$$

$$b_0^* \equiv -c(\tau/\tau) = -c. \tag{28}$$

As in the nonrelativistic situation, the exact location of the particle at any instant in time will be presumed not known. However, instead of specifying just the position probability density, which is not a relativistic invariant, we shall need a probability current density 4-vector j_μ, where j_0 is $-c$ multiplied by the probability density. The definition of the other components will be given later.

We now consider the problem of the "rest frame," which at first seems a perplexing situation, since the particle, continually moving with the speed of light, has no admissible rest frame in the usual sense. However, a useful definition may be given as follows: The *drift rest frame* is that Lorentz frame in which the one nonvanishing component of j_μ is $j_0 = -c\rho$. In this frame the stochastic particle will obey the Smoluchowski equation

$$\rho(\mathbf{x}, t+\tau) = \int \rho(\mathbf{x} - \lambda\hat{n}, t) P_R(\mathbf{x} - \lambda\hat{n}, t\,|\,\hat{n}(\Omega), \tau)\, d\Omega. \tag{29}$$

If we expand the left-hand side in a Taylor series about t, and the right-hand side in a Taylor series about x, we get

$$\rho(\mathbf{x}, t+\tau) = \rho(\mathbf{x}, t) + \tau \frac{\partial}{\partial t}\rho(\mathbf{x}, t) + \frac{\tau^2}{2}\frac{\partial^2}{\partial t^2}\rho(\mathbf{x}, t) \tag{30}$$

and

$$\begin{aligned}\rho(\mathbf{x}-\lambda\hat{n}, t)P_R(\mathbf{x}-\lambda\hat{n}, t\,|\,\hat{n}, \tau) &= \rho(\mathbf{x}, t)P_R(\mathbf{x}, t\,|\,\hat{n}, \tau) - \lambda P_R(\mathbf{x}, t\,|\,\hat{n}, \tau)(\hat{n}\cdot\nabla\rho(\mathbf{x}, t)) \\ &\quad - \lambda[\hat{n}\cdot\nabla P_R(\mathbf{x}, t\,|\,\hat{n}, \lambda)]\rho(\mathbf{x}, t) + \frac{\lambda^2}{2}\sum_{i=1}^{3}\Bigg[\left(n_i^2 P_R(\mathbf{x}, t\,|\,\hat{n}, \tau) \times \frac{\partial^2}{\partial x_i^2}\rho(\mathbf{x}, t)\right) + \left(n_i^2\rho(\mathbf{x}, t)\frac{\partial^2}{\partial x_i^2}P_R(\mathbf{x}, t\,|\,\hat{n}, \tau)\right) \\ &\quad + 2\left(n_i^2\frac{\partial}{\partial x_i}P_R(\mathbf{x}, t\,|\,\hat{n}, \tau)\frac{\partial}{\partial x_i}\rho(\mathbf{x}, t)\right)\Bigg] + O(\lambda^3).\end{aligned} \tag{31}$$

Using the relations

$$\int P_R(\mathbf{x}, t\,|\,\hat{n}, \tau)\, d\Omega = 1, \tag{32}$$

$$\int \lambda\hat{n}\frac{\partial}{\partial x}[P_R(\hat{x}, t\,|\,\hat{n}, \tau)]d\Omega = \lambda\frac{\partial}{\partial x}\langle\hat{n}\rangle = \tau\frac{\partial}{\partial x}\mathbf{b}, \tag{33}$$

$$\int \lambda^2\left(\sum_{i=1}^{3} n_i^2\right) P_R\, d\Omega = \int \lambda^2 P_R\, d\Omega = \lambda^2, \tag{34}$$

$$\int n_i^2 P_R\, d\Omega = \langle n_i^2\rangle = \tfrac{1}{3}\langle\hat{n}^2\rangle = \tfrac{1}{3}, \tag{35}$$

where the last equation ensures space isotropy for the hidden thermostat, we can now integrate (30) and (31) term by term to obtain (approximately)

$$\frac{\partial\rho}{\partial t} + \frac{\tau}{2}\frac{\partial^2}{\partial t^2}\rho = -\nabla\cdot(\rho\mathbf{b}) + \frac{\lambda^2}{6\tau}\nabla^2\rho. \tag{36}$$

If we substitute the values for τ and λ from (22) and (23) and make use of the 4-gradient $\partial^\mu \equiv (-\partial/\partial(ct), \nabla)$, (36) may be expressed in relativistic notation as

$$\partial^\mu(\rho b_\mu) - (\hbar/2m)\Box^2\rho = 0. \tag{37}$$

In the drift rest frame, the equivalent backward version of Smoluchowski's equation will also hold. Hence,

$$\rho(\mathbf{x}, t-\tau) = \int \rho(\mathbf{x}+\lambda\hat{n}, t)F(\mathbf{x}+\lambda\hat{n}, t\,|\,\hat{n}, \tau)\, d\Omega. \tag{38}$$

As analysis parallel to that given above for the forward diffusion situation yields the backward analog of (37),

$$\partial^\mu(\rho b_\mu^*) + (\hbar/2m)\Box^2\rho = 0. \tag{39}$$

If we define

$$j_b^\mu \equiv \rho b^\mu \tag{40}$$

and

$$j_{b*}^\mu \equiv \rho(b^*)^\mu \tag{41}$$

we can construct the total drift current

$$j^\mu \equiv \tfrac{1}{2}(j_b^\mu + j_{b*}^\mu) \tag{42}$$

and a drift velocity as

$$v^\mu \equiv j^\mu/\rho. \tag{43}$$

Then j_μ satisfies the continuity equation

$$\partial^\mu j_\mu = 0 \tag{44}$$

as can be seen by adding (37) and (39).

Equations (37) and (39) were derived on the assumption of a special Lorentz frame (the drift rest frame) and, as written, are not covariant with respect to Lorentz transformations since ρ is not a world scalar. However, we can define the term

$$(-1/c^2)j_\lambda j^\lambda \equiv |\rho|^2 \tag{45}$$

which is a world scalar, and generalize (37) and (39) to the covariant forms

$$\partial^\mu(j_b)_\mu - (\hbar/2m)\Box^2|\rho| = 0, \tag{46}$$

$$\partial^\mu(j_{b^*})_\mu + (\hbar/2m)\Box^2|\rho| = 0. \tag{47}$$

When $j_\mu = (-c\rho, 0, 0, 0)$, these reduce to the previous forms (37) and (39).

In order to construct the Klein—Gordon equation, it is necessary to adopt a definition for 4-acceleration, a_μ. Nelson's three-dimensional formula was

$$a(\mathbf{t}) = \tfrac{1}{2}\lim_{\Delta t\to 0}\left(\frac{\langle\mathbf{b}(t) - \mathbf{b}(t-\Delta t)\rangle^*}{\Delta t} + \frac{\langle\mathbf{b}^*(t+\Delta t) - \mathbf{b}^*(t)\rangle}{\Delta t}\right). \tag{48}$$

To extend this to four dimensions, we define a as follows:

$$\begin{aligned}a_\mu &\equiv \frac{1}{2}\left(\frac{\langle b_\mu(t) - b_\mu(t-\tau)\rangle^*}{\tau} + \frac{\langle b_\mu^*(t+\tau) - b_\mu^*(\tau)\rangle}{\tau}\right) \\ &= \tfrac{1}{2}(b_\lambda^*\partial^\lambda)b_\mu + \tfrac{1}{2}(b_\lambda\partial^\lambda)b_\mu^* + \frac{1}{2}\frac{\hbar}{2m}\Box^2(b_\mu - b_\mu^*),\end{aligned} \tag{49}$$

where Ito's rule for differentiation of forward and backward stochastic processes has been utilized as in the three-dimensional case. If the substitutions,

$$v_\mu = \tfrac{1}{2}(b_\mu + b_\mu^*), \tag{50}$$

$$\nu_\mu \equiv \tfrac{1}{2}(b_\mu - b_\mu^*) = (\hbar/2m)\partial_\mu \ln|\rho|, \tag{51}$$

are made, (49) can be written in the more tractable form

$$a_\mu = v_\lambda\partial^\lambda v_\mu - \nu_\lambda\partial^\lambda\nu_\mu - (\hbar/2m)\Box^2\nu_\mu. \tag{52}$$

For the fundamental dynamical rule, we adopt Einstein's relativistic version of Newton's law,

$$F_\mu = m a_\mu, \tag{53}$$

where F_μ is the force 4-vector. In particular, we shall consider only the Lorentz force F^μ associated with electromagnetic field $F^{\mu\nu}$, i.e.,

$$F^\mu = (e/c)F^{\mu\lambda}v_\lambda = (e/c)[\partial^\mu A^\lambda - \partial^\lambda A^\mu]v_\lambda, \tag{54}$$

where A^μ is the electromagnetic potential. The Lorentz gauge will be used so that

$$\partial_\mu A^\mu = 0. \tag{55}$$

The generalized momentum will be assumed to be derivable from the 4-gradient of the world scalar S, Hamilton's principal function of ordinary relativistic classical mechanics. Thus

$$\partial^\mu S = m v^\mu + (e/c)A^\mu. \tag{56}$$

If (52), (54), and (56) are substituted into (53), the result is

$$\left[\frac{1}{m}\left(\partial^\lambda S-\frac{e}{c}A^\lambda\right)\partial_\lambda\left(\partial^\mu S-\frac{e}{c}A^\mu\right)-mv^\lambda\partial_\lambda v^\mu-\frac{\hbar}{2}\Box^2 v^\mu\right]$$
$$=\frac{e}{c}\left[\partial^\mu A^\lambda-\partial^\lambda A^\mu\right]\left(\partial_\lambda S-\frac{e}{c}A_\lambda\right). \qquad (57)$$

If use is made of (55), this reduces to

$$\frac{1}{m}(\partial^\lambda S)\partial_\lambda(\partial^\mu S)-\frac{1}{m}\partial^\mu\left(\frac{e}{c}A^\lambda\partial_\lambda S\right)+\frac{1}{2m}\partial^\mu\left[\left(\frac{e}{c}\right)^2 A_\lambda A^\lambda\right]$$
$$=mv^\lambda\partial_\lambda v^\mu-(\hbar/2)\Box^2 v^\mu. \qquad (58)$$

Since both $\partial^\mu S$ and $v^\mu=(\hbar/2m)\partial^\mu \ln|\rho|$ are gradients of scalars, (58) may be written as

$$\partial^\mu\left[\frac{1}{2m}(\partial_\lambda S)(\partial^\lambda S)-\frac{e}{mc}A_\lambda\partial^\lambda S+\frac{1}{2m}\left(\frac{e}{c}\right)^2 A_\lambda A^\lambda\right.$$
$$\left.-\frac{m}{2}v_\lambda v^\lambda-\frac{\hbar^2}{4m}\Box^2\ln|\rho|\right]=0, \qquad (59)$$

or

$$\frac{1}{m}\left(\partial_\lambda S-\frac{e}{c}A_\lambda\right)^2+m(v_\lambda)^2-\frac{\hbar^2}{2m}\Box^2\ln|\rho|=M, \qquad (60)$$

where M is a constant. To determine the value of M, consider the classical limit case ($\hbar$ negligible) with vanishing vector potential A^μ. Then (60) becomes

$$(1/m)(\partial_\lambda S)^2=M \qquad (61)$$

or

$$mv_\lambda v^\lambda=M. \qquad (62)$$

Since in the drift rest frame

$$v_\lambda=(-c,0,0,0) \qquad (63)$$

it follows that

$$M=-mc^2. \qquad (64)$$

Thus the two basic equations of relativistic stochastic mechanics are (44) and (60). Both of these equations are of manifestly covariant form; all of the terms in them are either world scalars or, with the exception of j^μ and A^μ, the 4-gradients of world scalars. Moreover, since both j^μ and A^μ, when contracted with covariant 4-vector operator ∂^μ, form the world scalar zero, they must be covariant 4-vectors. Therefore the stochastic model satisfies all the necessary requirements of invariance with respect to Lorentz transformations.

To compare the stochastic relativistic model with orthodox quantum formalism, we recall the Klein–Gordon equation

$$\left(\partial^\mu-\frac{e}{c}iA^\mu\right)^2\psi-\left(\frac{m^2c^2}{\hbar^2}\right)\psi=0. \qquad (65)$$

Separating (65) into its real and imaginary parts, we obtain

$$\left(\partial^\mu S-\frac{e}{c}A^\mu\right)^2-\frac{\hbar^2}{R}\Box^2 R=-m^2c^2 \qquad (66)$$

and

$$\partial^\mu\left[R^2\left(\partial_\mu S'-\frac{e}{c}A_\mu\right)\right]=0, \qquad (67)$$

where R and S' are real functions which determine ψ through

$$\psi=R\exp(iS'/\hbar). \qquad (68)$$

The term $\Box^2 R/R^2$ obeys the vector identities

$$\frac{\Box^2 R}{R}=\frac{\Box^2 R^2}{2R^2}-\tfrac{1}{2}(\partial^\mu \ln R^2)^2 \qquad (69)$$

and

$$\Box^2(\ln R^2)=(1/R^2)\Box^2 R^2-(\partial^\mu \ln R^2)^2. \qquad (70)$$

By substituting (69) and (70) into (66) we get

$$\frac{1}{m}\left(\partial^\mu S'-\frac{e}{c}A^\mu\right)^2-m\left(\frac{\hbar}{2m}\partial^\mu\ln R^2\right)^2-\frac{\hbar^2}{2m}\Box^2\ln R^2=-mc^2. \qquad (71)$$

Now, if the identifications, $R^2=|\rho|$ and $S=S'$ are made, (67) and (71) become identical to (44) and (60), respectively. These stochastic results are not totally equivalent to the standard quantum formalism, since the Klein–Gordon equation allows both positive and negative energy eigenvalues. By contrast, in the stochastic model we have

$$E/c=p^0=mv^0\geq mc>0, \qquad (72)$$

and, thus, the stochastic derivation has been only for positive energies. However, negative energies can be easily incorporated into the theory if one redefines b_0 (and b_0^*) to be

$$b_0=-c[A(\mathbf{x},t)-B(\mathbf{x},t)], \qquad (73)$$

where $A(\mathbf{x},t)$ is the probability that the particle at $(\mathbf{x},t)$ will have positive energy during the time span τ, and $B(\mathbf{x},t)$ is the probability that it will have negative energy. A similar redefinition holds for b_0^* and hence for $v_0=\frac{1}{2}(b_0+b_0^*)$. Alternatively, one may adopt the interpretation that A is the probability the particle will go forward in time and B is the probability that it will go backward in time. Our derivation has thus far assumed $A=1$ and $B=0$. For a negative energy particle, the reverse would be the case.

Aron[3] and de la Pena-Auerbach[4] have also constructed derivations of the Klein–Gordon equation from stochastic concepts. Aron's approach yields two real equations which are not equivalent to the complex Klein–Gordon equation except in special cases.

De la Pena-Auerbach, by studying the properties of stochastic derivatives under time reversal, has arrived by a quite different theoretical route at equations essentially equivalent to our two basic relativistic stochastic equations. Surprisingly, he did not require the particle to have a fixed free speed c but did assume that

$$\frac{1}{2}\frac{\langle(\Delta x)^2\rangle}{\Delta t}=\frac{3\hbar}{2m}, \qquad (74)$$

where $\Delta t\approx\tau$.

However, it is easily demonstrable that (74) must be reduced to the more stringent requirement of our random-walk model in order to avoid inconsistency. Indeed if in a particular excursion, we had $|\Delta\mathbf{x}|>c\tau$, the world line of the particle would be spacelike and hence inadmissible. If, however, we always require $|\Delta\mathbf{x}| \leqslant c\tau$, then necessarily

$$\frac{1}{2}\frac{\langle(\Delta x)^2\rangle}{\tau} \leqslant \frac{1}{2}\frac{c^2\tau^2}{\tau} = \frac{3\hbar}{2m}. \tag{75}$$

The equality sign holds only if $|\Delta\mathbf{x}| = c\tau$, for all $\Delta\mathbf{x}$, which corresponds to our stipulation (20).

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