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QUANTUM THEORETICAL CONCEPTS OF MEASUREMENT*: PART I

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The overall purpose of this paper is to clarify the physical meaning and epistemological status of the term 'measurement' as used in quantum theory. After a review of the essential logical structure of quantum physics, Part I presents interpretive discussions contrasting the quantal concepts observable and ensemble with their classical ancestors along the lines of Margenau's latency theory. Against this background various popular ideas concerning the nature of quantum measurement are critically surveyed. The analysis reveals that, in addition to internal mathematical difficulties, all the so-called quantum theories of measurement are grounded in unjustifiable, *classical* presuppositions.

1. Measurement. Prior to the quantum era, the measurement concept was philosophically innocuous; it displayed a certain obviousness of meaning which occasioned little controversy. In fact, the postulates of classical theories made reference to it only implicitly. However, for reasons to be discussed below in connection with the related concept of *observable*, quantal propositions cannot suppress direct use of the term.

It is natural, therefore, in probing the quantum framework for logical inconsistencies, to seize upon this novel feature by demanding a consistent quantal description of the process of measurement; the replies given to this logical challenge are called *quantum theories of measurement*. Because measurement is not a concept in isolation, the study of such theories reveals diverse ideas concerning the nature of other quantal constructs. Hence the quantum theory of measurement offers a portal to philosophic understanding of the meaning and goals of quantum physics as a whole.

2. Mathematical foundations of quantum physics. To provide a firm basis for any analysis of measurement, it is essential to review the fundamental axioms of quantum physics. The postulates cited below attempt to capture the essential character of a *quantum* theory, not to enumerate every single mathematical assumption. This

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is standard practice in physics; we force an admittedly arbitrary distinction between unstated background postulates, which encompass much logic and mathematics, and *physical* postulates, which serve to delineate the rudiments of a particular branch of physics. Such a cleavage enables us to direct our philosophic inquiry more acutely to crucial physical points instead of detracting from that purpose by citing numerous minor axioms in the manner of the more tedious excursions into "quantum mathematics."

We now state and discuss three postulates which underlie all forms of modern quantum theory from wave mechanics to field theory, postulates which reflect the essence of the quantum approach to natural philosophy. From these we shall extract the primitive physical terms employed, and our analysis of measurement will then revolve about those basic constructs.

P1: (Correspondence Postulate) The[1] linear Hermitean operators, $A, B, \ldots$, on Hilbert space which have complete orthonormal sets of eigenvectors correspond to physical observables $\mathscr{A}, \mathscr{B}, \ldots$. The function $\mathscr{F}(A)$ corresponds to observable $\mathscr{F}(\mathscr{A})$ if A corresponds to $\mathscr{A}$.[2]

The correspondence postulate is often stated in the converse form: to every observable there corresponds an operator. However, we have shown elsewhere [21] that such a formulation leads to physically untenable consequences and must be rejected. The term 'Hilbert space' appears due to established physical usage; no stricture is intended on the application of newer mathematical constructs which may eventually provide the mathematical background for quantum theory.

P2: (Mean Value Postulate) To every ensemble of identically prepared systems there corresponds a real linear functional of the Hermitean operators, $m(A)$, such that if A corresponds to an observable $\mathscr{A}$, the value of $m(A)$ is the arithmetic mean of the results of $\mathscr{A}$-measurements performed on the member systems of the ensemble.

P3: (Dynamical Postulate) Every type of quantum system is dynamically characterized by a linear unitary operator T (the evolution operator) in the sense that the mean value functional $m_{t_2}(A)$ at time t_2 for an ensemble of such systems which at time t_1 had mean value functional $m_{t_1}(A)$ is given by

$$m_{t_2}(A) = m_{t_1}[T^{\dagger}(t_2, t_1)AT(t_2, t_1)].$$

P2 and P3 together indicate that the concept of physical state in quantum theory is represented by the mean value functional, the only quantal construct which relates to measurement results *and* obeys a causal law. Classically, this is perhaps the most objectionable feature of quantum theory, for $m(A)$ refers empirically only to an ensemble whereas a state representation traditionally belonged to individual

[1] To accommodate superselection rules [27], principles which in one form prohibit certain Hermitean operators from representing observables, the initial *the* in P1 might have to be replaced by *some* (cf. sec. 14).

[2] The observable $\mathscr{F}(\mathscr{A})$ is measured simply by measuring $\mathscr{A}$ and substituting the result a into the function $\mathscr{F}$; the range value, $\mathscr{F}(a)$, is then the result of the $\mathscr{F}(\mathscr{A})$-measurement.

systems in a nonstatistical sense. However, a cursory examination of these postulates does not immediately show that the older understanding of the state concept cannot somehow be extricated from them, although this is in fact the case [20]. We shall return to this point later.

To make contact with familiar elements of the quantum formalism, we next state a few key theorems which follow from P1–P3.

Th1[3]: For every mean value functional $m(A)$ there exists an Hermitean operator ρ such that

$$m(A) = \mathrm{Tr}(\rho A).$$

For mathematical convenience, it is fruitful to shift the emphasis from the functional $m(A)$ to the operator ρ related to it by Th1. Thus, the statistical properties of an ensemble are embodied in ρ, which is called the density operator.

Th2: $\mathrm{Tr}\ \rho = 1$.

Th3: The probability $W_{\mathscr{A}}(a_k; \rho)$ that an $\mathscr{A}$-measurement on a system from an ensemble with density operator ρ will yield eigenvalue a_k, $A\alpha_{kd_k} = a_k\alpha_{kd_k}$ is given by

$$W_{\mathscr{A}}(a_k; \rho) = \mathrm{Tr}(\rho P_{\mathscr{H}_k}),$$

where $P_{\mathscr{H}_k}$ is the projection operator onto the subspace $\mathscr{H}_k$ belonging to eigenvalue a_k: $P_{\mathscr{H}_k} \equiv \sum_{d_k} P\alpha_{kd_k}$.

Th4: The only possible results of $\mathscr{A}$-measurements are the eigenvalues of corresponding operator A.

Th5: The density operator ρ is positive semidefinite.

Of tremendous significance in the theory of measurement is the concept of ensemble *homogeneity*, emphasized by von Neumann [25]. For the present, we merely define it, deferring philosophic analysis to later sections.

Defn: An ensemble is said to be pure, or homogeneous, if every rearrangement and partitioning of member systems results in subensembles physically identical to the original. An ensemble which is not pure is said to be mixed, or a mixture.

In terms of the mean value functional, if $m(A)$ is pure, there do not exist $m_1(A)$, $m_2(A)$ satisfying $m(A) = w_1m_1(A) + w_2m_2(A)$, where w_1, w_2 are the fractions of the ensemble contained in the two subensembles; clearly, $w_1 + w_2 = 1$, $w_1 > 0$, $w_2 > 0$. In the language of density operators, we then have the following theorem:

Th6[4]: ρ is pure if and only if $\rho = P_\psi$, where P_ψ is the projection operator onto the span of Hilbert vector ψ. (ψ is usually called the "state" vector.)

[3] For proof, cf. [25], 313–316.

[4] For proof, cf. [25], 323.

Th7: For pure ensembles with state vector ψ, i.e. $\rho = P_\psi$,

$$m(A) = \frac{\langle \psi, A\psi \rangle}{\langle \psi, \psi \rangle},$$

where $\langle\ \rangle$ denotes scalar product. By convention, ψ is generally normalized so that $\langle \psi, \psi \rangle = 1$, hence $m(A) = \langle \psi, A\psi \rangle$.

The theorems above comprise the basic ingredients of quantum statics, so called because all statements essentially refer to a single instant of time. Quantum causality is embodied in the temporal development of $m(A)$ according to P3. From Th1, we have $m(A) = \mathrm{Tr}(\rho A)$; hence, temporal changes of the functional m may be represented by transformations either of the density operator (Schrödinger picture), or of the operators representing observables (Heisenberg picture), or of both. In what follows, quantum dynamics will be cast in the Schrödinger picture.

Th8: $\rho(t_2) = T(t_2, t_1)\rho(t_1)T^\dagger(t_2, t_1)$.
Th9: If $\rho(t_1) = P_{\psi(t_1)}$, then $\rho(t_2) = P_{\psi(t_2)}$, where $\psi(t_2) = T(t_2, t_1)\psi(t_1)$.[5]

Th7 and Th9 provide the justification for the common name "state" vector for ψ, since knowledge of ψ and T enables calculation of all measurement statistics for any instant.

3. Primitive terms of quantum theory. Scanning the foregoing postulates for those physical constructs which play major roles in general quantum theory, we find seven requiring careful study: system, preparation, ensemble, observable, measurement, result (of measurement), and state. It should be noticed immediately that none of these terms is intrinsically quantal; all of them have meaning, perhaps trivially in some cases, within the methodology of classical physics. However, within the quantal framework, some of them acquire extended significance and important subtleties of meaning.

Although our presentation of quantum theory mimics such rigorous mathematical systems as pure geometry by referring to primitives, postulates, and theorems, several distinctions must be recognized. When geometry is carefully axiomatized, the primitives are truly *undefined*; point, line, congruence, etc., are totally devoid of experiential meaning. Every relation among them is stated in the axioms, and these connections embody all properties to be associated with the terms. This information alone coupled with pure logic then leads to numerous initially hidden interrelationships among the terms, viz. the theorems. When the primitive concepts are provided with empirical counterparts via operational definitions the total scheme becomes physical geometry, the science of space. Ideally, perhaps every scientific discipline, including quantum theory, should be cast in this mathematically utopian form; but in fact even the relatively few physicists committed to logical rigor do not generally employ postulational schemes so pure as the rather exceptional case of geometry. Unfortunately, the construction of such

[5] If we formally define the Hamiltonian operator H by $T = \exp[(-i/\hbar)\int H\,dt]$, then $\psi(t)$ satisfies the *Schrödinger equation*, $H\psi(t) = i\hbar\,\partial\psi(t)/\partial t$.

mathematical systems is predicated upon considerable hindsight and is therefore inapplicable in the formative stages of a theory.

Returning now to quantum theory, we do not claim that the postulates of the last section embrace every relation among the seven primitives which might be invoked while deducing their consequences, nor can we assert that the selected list of basic terms is complete. Furthermore, none of these primitive constructs will ever be regarded as absolutely undefined; and in some cases their root physical definitions to be reviewed below will later demand further qualification. In spite of these departures from mathematical propriety, it is still possible to test quantum theory for logical consistency by following a program which parallels similar considerations in more rigorous logical systems.

Because the primitive constructs mentioned above are not *a priori* independent and undefined, it is necessary to begin with explanations which convey their *minimal* physical meanings. Such accounts will suffice until tensions in the logical matrix of primitives and postulates induced by the problem of measurement create the need for further explication.

The concept of *system* is understood throughout physics as the actual object of study; epistemologically, it is posited as the bearer of *observables* and hypostatized to become said object. A mathematician might be disposed to define a system as a set of observable-symbols, but such purity misses the point. An example of a quantum system is an electron in a given environment or, equivalently, a Dirac field in a single electron state interacting with other given physical systems. Since the concept of quantum observable is philosophically more sophisticated than its classical progenitor, we postpone the details to the next section. It is sufficient for the moment to state that whenever a system is subjected to the process called *measurement* of a given *observable*, there emerges a number, the *result* of the measurement. Thus observables serve to provide quantitative information about systems; every observable is endowed with measurement procedures which, if performed upon the system, yield the numerical results. Accordingly, observables are also called physical quantities. For reasons to be discussed later, we have purposely described these classically transparent concepts in what seems at first to be an overly cumbersome manner.

In accordance with the emphasis in physics on reproducibility of phenomena, a single measurement carries little significance. Systematic study of a given type of system therefore requires a well-defined, repeatable process of *preparation.* In general, what is of interest in physics is a set of measurement results for several observables, where the measurements are all performed upon *identically prepared systems.* Since acts of preparation are themselves physical processes under the governance of quantum theory, an interesting exercise related to the theory of measurement is the quantal description of a preparation. We shall return to this idea subsequently (section **14**, Part II).

The collection of identically prepared systems upon which the various measurements are performed is called the *ensemble*; more than one philosophic stand has been taken by physicists regarding the exact status of the quantum ensemble. It turns out that the different requirements placed upon the measurement act depend strongly on different meanings attached to the ensemble concept and the related construct, physical *state*. A later section will contrast the various kinds of ensembles used in physics in order to identify the rather unique character of the quantum ensemble.

In all physical theories, the *state* of a system refers to its momentary physical condition; it is the seat of causality in physics in the sense that some law of motion controls its temporal evolution. By "physical condition" is meant that states are related somehow to observables, and to measurement results. In the form given above the quantal postulates seem to correlate the state concept to a system only through an intervening ensemble of such systems identically prepared. Only the *statistics* of measurement results obey a causal law. Thus, in effect quantum theory seems to shift the reference of the state concept from the individual physical system to the ensemble.

It might be objected that this modification is illusory, that the postulates were stated with a distorted emphasis on ensembles which hides the true meaning of state. Thus, classical statistical mechanics might be axiomatized in a similar format; but the classical individual state would be lurking in the shadows and could be exposed with the proper logical illumination. Elsewhere [20] we have carefully examined this question and demonstrated that, while such is indeed the case for classical statistics, no such analogous reduction to individual states is possible within the quantum framework.

A quantum state refers *irreducibly* to an ensemble; an ensemble is defined by its mode of preparation and characterized by the statistics of measurements performed upon its member systems, and these statistics determine the state. Thus it is often convenient to speak of a *preparation of state*, a concept emphasized by Margenau [18] to delineate a class of physical processes often erroneously called measurements. This completes our preliminary survey of the key terms of general quantum theory; but before attempting to describe quantally the process of measurement, we shall undertake deeper analysis of the constructs *observable* and *ensemble*.

4. The nature of quantum observables. To those physicists who take mathematics to be in the same category as metal-working lathes and vacuum pumps, the subtleties taught by modern mathematicians often seem inane. Among these is the difference between a function f and its range value at domain point x, $f(x)$. Even in classical mechanics, however, there are two instances in which this distinction is physically meaningful, for it represents a philosophically important dichotomy among physical constructs. Consider first the case of the Hamiltonian H, a function of phase (q, p). Here the value of logically distinguishing H and $H(q, p)$ is eventually recognized by anyone thoughtfully studying analytical mechanics. In fact, the term *functional form* is often used to stress that the function itself, not its value, is under consideration. H itself is the mathematical representative of the system of interest; i.e. "the functional form of $H(q, p)$" contains the dynamical characteristics of the system and represents it in the law of motion. The numerical value of $H(q, p)$, on the other hand, is usually just the result which would be obtained if an energy measurement were performed on the system. (For the sake of familiarization, we continue to use this tedious phraseology introduced earlier for *minimal* descriptions of measurement.)

Failure to take note of the difference between H and $H(q, p)$ can actually lead to faulty reasoning of physical significance. For example, consider a mechanical problem with initial conditions $H(q, p) = E$ and $q = q_0$ (E and q_0 are constants). Hamilton's equations, used properly, will determine the motion. But consider the following reasoning: since $H(q, p)$ is not explicitly time-dependent, $H(q, p) = E$ not just initially but throughout the motion, according to a basic theorem. Hamilton's equations therefore become especially simple if the constant E is substituted for H. The immediate results are

$$\dot{p} = -\frac{\partial H}{\partial q} = -\frac{\partial E}{\partial q} = 0, \qquad \dot{q} = \frac{\partial H}{\partial p} = \frac{\partial E}{\partial p} = 0;$$

hence $q(t) = q_0$, $p(t) = 0$. The particle just sits still! That this solution is wrong is easily seen by considering $H = (p^2/2m) + (k/2)q^2$, the simple harmonic oscillator, which under initial conditions of the type given is not in general immobile. The error lay of course in neglecting the distinction between a function and its value.

There is a second instance in classical mechanics where this mathematical point could be stressed; it was not, however, until the advent of quantum mechanics that its message became apparent. Because of its dynamical significance, the function H is rather special; but in classical mechanics, every function of phase has physical meaning. A function corresponds to an observable; and the value of a function for a state (q, p) is, again in our "minimal" phraseology, the result which would be obtained if a measurement of the observable were performed upon a system in said state. Since the state *uniquely* determines the measurement result for every observable through the corresponding function, the natural classical manner of describing the situation was not a minimal account but rather the simpler assertion that in a state (q, p) the system had an observable A of value $A(q, p)$. For example, "the oscillator has an energy of 30 ergs"; and, *of course*, if an energy measurement is performed, the result would be 30 ergs—but to state this explicitly seems pointless. Thus with the notable exception of H, classical mechanics did not require rending the function from its value, nor the observable from its measurement result; and the concept of measurement entered only implicitly into physical discourse.

A glance at the postulates and theorems in section **2** suggests that no such departure from the minimal terminology is admissible in quantum theory. There the constructs observable and measurement result are related only via *probabilistic* connections, and *measurement* thereby emerges as a construct which must appear *explicitly* in quantal propositions.

This separation of the concepts observable and measurement result is of considerable importance to the philosophic understanding of quantum physics. The peculiar nature of quantal observables has been depicted in several ways, three of which we shall briefly review[6]: Bohr's complementarity principle [4], Margenau's latency theory [17], and Heisenberg's "potentia" doctrine [12].

Complementarity is accorded at least token recognition in virtually every introductory quantum text. Its basic premise is apparently that the nature and results of

[6] Outside the present context, the divergence of these three viewpoints far exceeds their similarity, as later sections will illustrate.

microphysical research demonstrate that what we called a minimal account is also a maximal account. The utter impossibility of direct perception of atomic objects suggests the separation of observable and measurement result; the failure of all attempts to preassign unique measurement results to all observables by careful preparation of state renders the separation final. Thus, given an atomic object (including a mode of preparation), there is the choice of measuring any observable and a theory which provides probabilities for the possible results; but to say that the system has position q_0, momentum p_0, energy E_0, etc. is physically meaningless. Accordingly, Bohr coined the term *complementary* to describe this characteristically quantum relationship among the observables.

Margenau's latency theory classifies observables by the terms *possessed* and *latent*. In the classical proposition that a particle has a certain energy, the energy is clearly understood as a property *possessed* by the system. Similarly, any classical function of state—mechanical, electrodynamic, or thermodynamic—denotes an observable attached possessively to a system. Nevertheless, in classical physics there are also observables associated with systems not possessively but responsively; i.e. if subjected to a certain environment, a system displays a property not constantly exhibited. For example, the acoustic "observable" pitch is inapplicable to a vibrating reed in an evacuated box; but if the enclosure is opened to the atmosphere, a "value" of the pitch emerges. Such is the nature of a *latent* observable. Because it is impossible to assign values to all the observables of a quantum system in a possessive way and because quantum theory unavoidably deals only with statistics of measurement results, most of its observables are latent. It is strictly improper to speak of a quantum system's *having* energy E_0; the strongest admissible statement is the conditional one that, if an energy measurement were performed, the result would, with some calculable probability, be E_0. Quantum observables are thus latent in the sense that their values appear only in response to measurement. Quantal latency for a given observable is represented mathematically by the existence of physically realizable state vectors which are not eigenvectors of the corresponding operator, i.e. states for which measurement results for the observable in question are irreducibly unpredictable. Hence, almost all quantal operators correspond to *latent* observables.[7]

Another description of the nature of quantum observables appears in Heisenberg's discussions of the Copenhagen interpretation. The state of a quantum system before measurement is envisaged as a set of tendencies likened to Aristotelian *potentia*. Upon measurement one possibility is fulfilled, as an actual, perceptible event occurs, culminating in the extraction of a number. Measurement of an observable is thus depicted as a "transition from the possible to the actual," which is but another way to state the latent character of the construct observable in quantum theory.

5. The nature and purpose of ensembles. In any physical theory which assigns probabilities to possible measurement results, use of the construct *ensemble* is un-

[7] Perhaps those which generate superselection rules are an exception: in one form, superselection rules exclude pure states which are not eigenstates of certain observables (cf. sec. 14, Part II); therefore it would always be possible to regard such observables as possessed.

avoidable, simply because probability in physics means relative frequency. This is not to say, in sterile operationist fashion, that through this empirical definition probability acquires its total significance. The situation in probability theory is no different in this respect from the rest of physics; i.e. constructs are endowed not only with empirical definitions but also with theoretical ones. In the case of probability, the theoretical side has long been in controversy; rival mathematical schools lay the foundations in different ways. As with other mathematical choices, physicists adopt the most naively intuitable version capable of meeting their needs. However, in any case probability, as drawn from mathematics, is logically a primitive term defined implicitly by the axioms in which it is embedded. It obtains physical meaning only when the epistemic rule of correspondence[8] is invoked which correlates it to the relative frequency of measurement results.

Although the foregoing remarks correctly portray the epistemological status of physical probability, the historical development of course did not proceed so logically. As Carnap [6] has noted, the search for a good theoretical definition of probability is a problem of explication, i.e. replacement of an old, vague concept by a new, exact one. Thus the logically prior mathematical theory was itself inspired by common-sense notions of probability as a measure of tendencies or propensities for events to occur. Undoubtedly such ideas likewise underlie—and perhaps undermine—the physicist's conception of probability.

Consider, for example, the following proposition: if a measurement of observable $\mathscr{A}$ is performed upon a system (prepared in a specified manner), the probability of obtaining the result a_1 is W_1. Superficially, layman and physicist alike construe W_1 as somehow reflecting a tendency for the emergence of a_1 from that system at the instant of measurement; but careful consideration reveals the rather mystical tenor of that view. Actually W_1 is a clearly defined quantity, viz. the relative frequency of the result a_1 arising from $\mathscr{A}$-measurements upon an ensemble of identical systems $\mathscr{S}$ all prepared in the manner Π. Whatever physical information about the pair $(\mathscr{S}, \Pi)$ probabilities like W_1 may carry, the connection between system and probability is always through the intermediary construct, ensemble; otherwise, probability is a concept too hazy to qualify for a place in physics.

So far we have discussed only the precise meaning of physical probability; that analysis now justifies a shift in emphasis from probability itself to the intimately related notion of ensemble, since a probability without an ensemble is unphysical. Of special importance, the various ways in which probabilities enter theoretical physics are mirrored in the nature and purpose of the associated ensembles. The relevance of a study of classical and quantal ensembles to the quantum theory of measurement will become clear in later sections.

A physical ensemble is basically a set of identically prepared independent systems. In principle, measurements can be performed on the constituents at any instant after preparation; and the set is sufficiently large to warrant statistical analysis of the measurement results, including meaningful identification of probability as relative frequency. However, the phrase, "a set of identically prepared systems,"

[8] For elaboration, cf. [17].

represents an abstraction physically realizable in several ways. The term "set" denotes a *mental* collection of objects which need not even coexist; a set may be an aggregation of elements all present at once, a temporal sequence of single elements, or any admixture of these two extremes. Similarly, "identically prepared systems" might refer to just one system prepared and sequentially reprepared. Whichever combination is selected, the member systems are strictly independent; for example, the assemblage of molecules constituting a real gas is not an *ensemble* of molecules.

The physical significance of an ensemble depends not just on its structure but also on its purpose, i.e. on the connection between the ensemble and the actual physical situation to which it refers. In the classical realm, perhaps the simplest ensemble imaginable is a collection of coexisting, noninteracting mechanical systems. If their common preparation process consists of placing a system in a given dynamical state, the resultant ensemble will then be *homogeneous*, or *pure*, for obviously every subensemble is identical to the whole ensemble so far as measurement statistics are concerned. However, if the mode of preparation is less discriminating, there will be a distribution of states over the ensemble, sets of measurement statistics will vary among subensembles, and hence the whole ensemble will be *mixed*. Although this simple ensemble thus illuminates the basic physical meaning of ensemble homogeneity, defined mathematically in section **2**, it leaves the impression that the homogeneity concept is all too trivial to be of any value. However, this seeming triviality is but a manifestation of the classical context in which the example was given; in particular, it was implicit in the language used that the observables, hence the classical states, were *possessed*, a property which enables a convenient *pictorial* conception of the systems.

In Gibbsian statistical mechanics, an ensemble of the type just described is employed; but it is not used directly, i.e. the physical system of interest is not itself a collection of coexisting, noninteracting, identically prepared systems. In fact, it is just one such system, related to the imaginary ensemble of replicas by a postulated correspondence between observed values and ensemble averages. Why, then, is an ensemble used at all? From a strict mechanistic viewpoint, the reason might be simply that thermodynamic systems, whose behaviour Gibbs sought to comprehend mechanically, are incredibly complex. Actual knowledge of a precise mechanical state for the septillion molecules in a mole of gas is a practical chimera. Gibbs' virtual ensemble could be regarded, therefore, as a mathematical representation of such *ignorance*. In fact, the scheme was later generalized to become modern information theory. It should be stressed, however, that we have asserted only that the Gibbsian ensemble *permits* consistent interpretation in terms of ignorance, not that it *must* be so understood. Indeed, so long as the physical significance of ergodic theory remains in dispute, there is a possibility that even a complete mechanical state specification of a complex system would not account for its thermodynamic behavior, in which case the Gibbsian ensemble would be a physical construct far more abstract and fundamental than its "ignorance interpretation" suggests. Gibb's ensemble allows the ignorance interpretation chiefly because it is framed within a classical metaphysic which provides something to be ignorant of, viz. the values of possessed observables.

Consider now the transition to the latent observables of quantum theory and its impact upon everything said about classical ensembles. That simple ensemble consisting of a simultaneous assembly of noninteracting systems identically prepared now requires a "minimal" description. Strict emphasis on measurement results and the correlation of their statistics to modes of preparation replaces the graphic account in terms of individual classical states. The first preparation instruction given above, "place each system in a given dynamical state," is now quite meaningless. In view of the latency of quantal observables, the most that can be said is to "use a method of preparation such that the associated statistics of measurement results indicate a homogeneous ensemble." Similarly, some preparation schemes will produce ensembles whose measurement statistics are summarized by an inhomogeneous mean value functional. The essential point is that this latency-enforced revision of ideas destroys the basis for interpreting ensembles as expressive of ignorance in the Gibbsian sense, for *in quantum physics there are no longer even in principle any innate quantities of which to be "ignorant."* In quantum theory the actual object of study is effectively the ensemble itself; however, that ensemble may be any of the types described earlier in this section. In particular, it might even be a single quantum system in a temporal alternating sequence of identical preparations and diverse measurement operations [18].

As mentioned earlier, we have demonstrated elsewhere that, by contrast to the classical case, in quantum theory the concept of homogeneity cannot be used in a consistent way to assign physical states to *single* elements of an ensemble. Although ordinary physical jargon speaks of "a system in the state ψ," that phrase can only mean either (1) an element of a pure ensemble with density operator $\rho = P_\psi$, or (2) an element of a pure subensemble ($\rho^{(1)}$) of a general mixed ensemble whose density operator *may* be expanded as $\rho = w_1\rho^{(1)} + w_2\rho^{(2)}$, $w_1 + w_2 = 1$, $w_1 > 0$, $w_2 > 0$, $\rho^{(1)} = P_\psi$. The fact that in case (2) the *same* element may equally well be called "a system in (another) state φ" (since the expansion of ρ into pure subensembles is not unique) proves the absurdity of literally associating a state vector with a single member of an ensemble. (For elaboration, see reference [20].)

Nevertheless, because any mixed ensemble can in principle be subdivided into sets of pure subensembles, there is a logically weak sense in which the quantum mixture is often linked to ignorance: by analogy to the Gibbsian case, the mixed ensemble is sometimes interpreted to mean that there is ignorance as to which pure state the system is "really in." Indeed in the discipline called quantum statistical mechanics, this fiction is artificially upheld by conjuring up "two averages" from the quantal mean value expression $m(A) = \mathrm{Tr}(\rho A)$. Suppose $\rho = \sum_k w_k P_{\psi_k}$ is one among the *many* ways the ensemble at hand can be grouped into *pure* subensembles, w_k being the fraction of the original ensemble in the P_{ψ_k}-subensemble, if this particular selection is made. Then, $m(A) = \mathrm{Tr}(\rho A) = \sum_k w_k\langle\psi_k, A\psi_k\rangle$ is the average result of $\mathscr{A}$-measurements on the ensemble with density operator ρ. Now, despite the fact that this expansion is not unique, it is standard practice in statistical mechanics to declare, as for example ter Haar [24] does, that $m(A)$ "is twice an average. First we take the quantum mechanical average . . . in a system described by the wave function ψ_k, and, secondly, we take the average over the ensemble." The introduction to the same chapter typically cautions that in quantum ensemble theory, "one must be extremely careful to make a clear distinction between the statistical aspects inherent in quantum mechanics and the statistical aspects introduced by the ensembles." Such statements sound as though (1) ψ_k-statistics are not related to ensembles, and (2) that *mixed* density operators always refer to ensembles made up of systems "really in" pure states ψ_k.

Actually, (1) and (2) are both false; however, if we make the spurious identification of ψ_k as the quantal analogue to a classical state (as noted above, there is none) and interpret "ensemble" in (2) as meaning "virtual assembly of coexisting, identically prepared systems," there results a pseudo-analogue to Gibbs' method which is of motivational value in quantum statistical mechanics. To make the analogy complete, there must be a postulated connection between $\mathrm{Tr}(\rho A)$ and observed values of thermodynamic quantities. Such a postulate together with knowledge of specific ρ's is essentially the logical core of quantum statistical mechanics; meaningless "classical" analysis of quantum ensembles is not necessary, although it may serve to *suggest* the formulation of useful ρ's. However, its intuitive value in this context should not be mistaken for rational physics.

This digression on quantum statistical mechanics was not made to condemn its heuristic methods but to repudiate the erroneous idea that the general density operator represents ignorance in perfect analogy to the Gibbsian model; the density operator is not at all the sole property of quantum statistical mechanics but is actually a basic quantal construct. In fact, a mixed ρ cannot refer to an ensemble of systems each "really in" a pure state since, as we have repeatedly emphasized, that phraseology is logically ambiguous. A mathematically parallel situation in classical optics arises for polarization of light. If a light beam is, for example, unpolarized, we cannot meaningfully conclude that there are "really" two incoherent "sub-beams" of equal weight each linearly polarized but along perpendicular directions, for the analysis is not unique. With equal justification, many other such dissections of the unpolarized beam may be performed, among these the assertion that the "sub-beams" are "actually" circularly polarized in opposite senses. Empirically, every unpolarized beam can be split *either* way with the proper equipment; thus propositions about the "hidden structure" of the unpolarized beam are physically meaningless. Just as there are light beams which are intrinsically unpolarized or partially polarized, there are quantum ensembles which are intrinsically mixed; neither has anything to do with ignorance.

To complete this discussion of ensembles, we draw attention to a striking difference between the classical and quantal cases. A classical ensemble is described by the set of probabilities that member systems are in the various pure subensembles. Because the latter correspond to classical states, they are not only statistically homogeneous (as defined in section **2**) but also dispersionless, which means that for any observable, measurement results from a pure subensemble are all identical. The collection of pure subensembles into which any given ensemble may be resolved is unique. Because it is physically possible to select the unique, pure, dispersionless subensembles from the original ensemble, the above mentioned probabilities may be called *reducible*, a term used in this connection by Margenau [19].

For a quantum ensemble, the reduction to pure subensembles is no longer unique; nevertheless, similar selection processes are still possible. Once a resolution to homogeneous subensembles has been specified, the total ensemble is then characterized in part by reducible probabilities just as in the classical case. However, by a theorem of von Neumann, no quantum ensemble, not even a homogeneous one, is dispersionless.[9] Therefore, reduction to pure subensembles does not explain away all probabilities; there always remain probabilities, called *irreducible* by Margenau [19], which reflect the intrinsic dispersion of homogeneous quantum ensembles. Incidentally, this property is the backbone of Heisenberg's Principle of Indeterminacy. To summarize: a classical ensemble may be reduced to a unique set of homogeneous, dispersionless subensembles; a quantum ensemble may be reduced to any one of numerous sets of homogeneous subensembles each of which invariably exhibits dispersion in the statistics of measurement results for most observables.

6. Measurement in the Copenhagen interpretation. Throughout the literature on quantum measurement, one key point is accorded universal acceptance as the fundamental desideratum of a quantum theory of measurement, viz. the proposition that $\mathscr{A}$-measurements upon an ensemble whose initial density operator is $\rho^{(n)} = P_{\psi_n}$, $\psi_n = \sum_k \langle\alpha_k, \psi_n\rangle\alpha_k$, where $\{\alpha_k\}$ are the eigenvectors of $\mathscr{A}$'s operator A, will produce a post-measurement ensemble whose density operator is

$$\hat{\rho}^{(n)} = \sum_k |\langle\alpha_k, \psi_n\rangle|^2 P_{\alpha_k}.$$

From this it follows that $\mathscr{A}$-measurements upon an initially mixed ensemble characterized by $\rho = \sum_n w_n\rho^{(n)}$ induce this transformation in the density operator:

$$\rho = \sum_n w_n\rho^{(n)} \to \hat{\rho} = \sum_n w_n\hat{\rho}^{(n)} = \sum_n w_n \sum_k \langle\alpha_k, P_{\psi_n}\alpha_k\rangle P_{\alpha_k}.$$

This may be expressed as

$$\rho \to \hat{\rho} = \sum_k \langle\alpha_k, \rho\alpha_k\rangle P_{\alpha_k},$$

the form used by von Neumann; or if P_{α_k} is written $|\alpha_k\rangle\langle\alpha_k|$ (Dirac notation), this "$\mathscr{A}$-measurement transformation" assumes another simple form,

$$\rho \to \hat{\rho} = \sum_k P_{\alpha_k}\rho P_{\alpha_k}.$$

Henceforth we shall frequently refer to $\rho \to \hat{\rho}$ simply as *the* (von Neumann) *measurement transformation.*

Thus the philosophical challenge of the measurement concept in quantum theory is generally translated into mathematical physics as follows: prove that the measurement interaction of a system with an $\mathscr{A}$-meter transforms the system density operator in the manner just defined. Sometimes this problem is expressed in the colorful language of waves: prove that measurement "destroys coherence" or

[9] For proof, cf. [25], 332.

"introduces random phase relations." The origin of these phrases is to be found in the historic analogy between quantum mechanics and classical optics, which is especially clear in the familiar Schrödinger wave mechanics.

Unfortunately, such picturesque analogies to classical optics can never deepen our understanding of quantum measurement; indeed, they may even becloud the real issues. Although it is undeniable that quantum theoretical *calculations* often bear striking resemblance to those of classical optics and acoustics (for obvious historical reasons) and that working physicists therefore draw heavily on such *mathematical* parallels in the course of everyday problem solving, nevertheless, as *physical* theories, wave optics and "wave" mechanics are strikingly distinct. As a consequence of this physical gulf, these mathematical analogies may offer more confusion that illumination when applied to a problem so fundamental as the quantum theory of measurement.

The reason that proof of the measurement transformation is so commonly accepted as the goal of quantum measurement theory is undoubtedly the compatibility of that goal with the so-called Copenhagen interpretation of quantum theory to which we now turn for an explanation.

Because of the explicitness of the proposition which we are asking the Copenhagen school to justify, general epistemological considerations like those of Bohr and von Weiszäcker are not too helpful. On the other hand, Heisenberg tends to be more specific in his philosophic discussions and has in fact given detailed expositions of the nature of measurement; accordingly, we shall take him as spokesman for the so-called[10] Copenhagen version of quantum measurement theory.

A common method of elucidating a complex subject is by analogy to something familiar, provided the analogy is not too superficial or purely poetic. Thus we have seen in previous sections that features of quantum theory can be explained by drawing parallels to both classical optics and statistical mechanics. In the case of optics, the possibility of confusion loomed large; but the framework of statistical mechanics provided an excellent analogue to that of quantum theory—up to a point. Heisenberg has expounded the Copenhagen version of measurement theory by appealing to the latter analogy. Because of its evident impact upon quantum philosophers and theorists, we now closely scrutinize his analysis.

With Heisenberg [11] consider, first from the standpoint of classical Gibbsian statistics, a hot metal occasionally emitting a thermal electron. Near this emitter is a photographic plate which registers all electrons emitted above some established threshold velocity. The temperature T of the metal is measured, and its thermodynamic state is represented mechanically by the canonical ensemble, i.e. $\rho(q, p) \propto \exp(-H(q, p)/kT)$, where H is the Hamiltonian of the metal. Now, as time passes, $\rho(q, p)$ develops in accordance with Liouville's equation; in particular, if the composite system of metal plus plate is considered, it is in principle possible

[10] It might well be argued that there are many "Copenhagen interpretations" and that the present section deals with the *Heisenberg* "Copenhagen interpretation" as opposed, for example, to the Bohr "Copenhagen interpretation"; however, we shall not enter into that debate. In the present context, the term "Copenhagen interpretation" will be used in the same way Heisenberg uses it.

to compute the probability that a given number of electrons have been detected by a certain time. Now, as noted in the preceding section on ensembles, it is possible to regard this use of the canonical ensemble as an expression of mechanical *ignorance*. Heisenberg clearly takes this position when he says that if an "observer is present, he will suddenly register the fact that the plate is blackened. The transition from the possible to the actual is thereby completed as far as he is concerned; he correspondingly alters the mathematical representation discontinuously, and the new ensemble contains only the blackened photographic plate. . . . We see from this that the characterization of a system by an ensemble not only specifies the properties of this system, *but also contains information*[11] *about the extent of the observer's knowledge of the system.*"

To complete this classical analogue to the Copenhagen version of quantum measurement, it is necessary to provide a counterpart to complementarity. Following an idea of Bohr [5], this may be done by recalling from statistical thermodynamics that a closed system is properly represented by a microcanonical ensemble $\rho(q, p) \propto \delta(H(q, p) - E_0)$, whereas an open system (in thermal equilibrium with a heat reservoir) requires a canonical ensemble, $\rho(q, p) \propto \exp(-H(q, p)/kT)$. In the former case, the energy is fixed but the temperature is not determined; to measure the temperature, the system must be "opened" and put in thermal equilibrium with a thermometer. But when that is done, the energy fluctuates in accordance with the canonical distribution. Thus a macroscopic description involving the concept temperature is more or less "complementary" to a precise micromechanical description in which temperature is undetermined.

Obviously applying this "classical complementarity" to the hot metal and photographic plate, Heisenberg reasons that complete knowledge of the microstate of a *closed* metal-plus-plate system would permit exact rather than just probabilistic predictions concerning the blackening sequence, but "the statement of the temperature would then have been completely meaningless." On the other hand, if that composite system is open to its environment (called by Heisenberg "the external world"), then temperature supposedly becomes meaningful but precise knowledge of the microstate no longer eliminates probabilities, exact prediction being precluded because "we do not know every detail of the external world."

The Copenhagen interpretation is essentially an attempt to provide exact quantal analogues for the concepts of statistical thermodynamics, provided the latter are understood in ways just explained.[12] A logical first step would be to determine (1) what in statistical mechanics corresponds to the density operator, and (2) to what extent the analogy is correct. As we have already pointed out, as a mathematical object characterizing statistics of measurement results for an ensemble, the density operator plays the same role as Gibbs' density-of-phase; moreover, a quantum ensemble having a state vector (i.e. $\rho = P_\psi$) is analogous—so far as *homogeneity* is concerned—to a classical ensemble of systems all in the same mechanical state. But we have also observed that state vectors differ from classical microstates in ways (to

[11] None of the italics in the quotations of this section is in Heisenberg's original papers.
[12] It should be noted that several of the above statements from classical statistics as interpreted by Heisenberg are familiar but not universally agreed upon by theoretical physicists.

be recalled as needed) which render this last analogy imperfect. Copenhagen theorists tend to ignore these infelicities; thus Heisenberg carries the ignorance interpretation of classical mixed ensembles over to quantum theory when he explains[13] that "the probability function combines objective and subjective elements" with this exception: "In ideal cases, the subjective element in the probability function may be practically negligible as compared with the objective one. The physicists then speak of a 'pure case'." We have already called attention in section **5** to the inconsistency of this viewpoint. To regard a quantal mixture as expressing subjective ignorance of actual objective pure states is, in view of the essential latency of quantum observables, physically meaningless. Any attempt to assign pure states to individual elements of a mixed ensemble encounters hopeless ambiguity [20], primarily because of a deep logical fissure in the analogy to classical statistics, viz. the circumstance that in quantum theory homogeneity does not eliminate dispersion. The Copenhagen interpretation therefore pushes the analogy between density operator and density-of-phase beyond its proper bounds.

A second quantal analogue to statistical mechanics is based on the effects of interaction. We have systematically contrasted the dynamics of classical and quantal interactions elsewhere [20]; superficially the analogy seems a good one, for in both theories an initially pure ensemble evolves into a mixture upon interaction with a *mixed* ensemble. In accordance with the ignorance interpretation of ensembles, Heisenberg therefore asserts that a system open to the "external world" must be described by a mixed ensemble, "*since we do not know the details of the 'external world system'*." [11] This reasoning is correct in classical physics but fallacious in quantum theory. Indeed, in the latter case, even if the "details were known" so that no "subjective" element entered the description of the "external world," i.e. even if the "external world" were in an objective, *pure* state, still the initially pure open system would evolve into a mixture! Once again, quantum theory proves incompatible with the ignorance interpretation of ensembles.

In any case, only a *closed*[14] system can be dynamically characterized by a state vector; thus just as temperature was declared "meaningless" for a closed classical system, so apparently are all physical quantities for a closed quantum system. As Heisenberg puts it [11], although state vectors are objective, they are "abstract and incomprehensible," and "do not refer to real space or to a real property." To make an actual measurement, system-plus-apparatus must be open, for "connection with the external world is one of the necessary conditions for the measuring apparatus to perform its function." It follows of course that system-plus-apparatus can only be an element of a mixed ensemble; and for Heisenberg this automatically entails "statements about the observer's knowledge. If the observer later registers a certain behavior of the measuring apparatus as actual, he thereby alters the mathematical representation discontinuously, because a certain one among the various possibilities has proved to be the real one."

In mathematical terms, the Copenhagen description of an $\mathscr{A}$-measurement on a

[13] [12], 53.

[14] By *closed* we mean a system not interacting with its environment; i.e. the Hamiltonian for system plus environment has no interaction term.

system $\mathscr{S}$ from a pure ensemble therefore runs as follows. $\mathscr{S}$ is initially closed and in the state $\psi = \sum_k \langle \alpha_k, \psi \rangle \alpha_k$, which represents objective tendencies toward the possible $\mathscr{A}$-values $\{a_k\}$ with respective probabilities $\{|\langle \alpha_k, \psi \rangle|^2\}$. However, since $\mathscr{S}$ is isolated, no $\mathscr{A}$-measurement can be performed; for measurement requires interaction with surroundings. Now, an open system must be described by a density operator, i.e. by a whole ensemble of systems in various states. When $\mathscr{S}$ is "opened" for an $\mathscr{A}$-measurement, its proper representative is therefore such an ensemble; but this introduces a subjective element, viz. *ignorance* as to which of the $\mathscr{A}$-values actually obtains. The density operator after the measurement interaction (but before the actual $\mathscr{A}$-value is observed) is accordingly $\hat{\rho} = \sum_k |\langle \alpha_k, \psi \rangle|^2 P_{\alpha_k}$, since a system in eigenstate α_k is certain to have the $\mathscr{A}$-value a_k, and $|\langle \alpha_k, \psi \rangle|^2$ is just the probability originally associated with that value. Finally, observation of the actual $\mathscr{A}$-value eradicates the ignorance; and, if a_k is the result, the state α_k is assigned to the system. The overall effect of an $\mathscr{A}$-measurement upon the state $\psi = \sum_k \langle \alpha_k, \psi \rangle \alpha_k$ was therefore contraction of ψ to one term α_k, an act called by Copenhagen theorists "reduction of the wave packet."

There is the official answer to our question as to why derivation of the measurement transformation is popularly adopted as the goal of quantum measurement theory. That transformation, originally formulated by von Neumann, is indeed the correct mathematization of the Copenhagen philosophy of measurement. If the latter were espoused, the postulates of section **2** would have to be augmented by the following common statement connecting the measurement concept to wave packet reduction in a definitive, analytic sense:

P: (*Strong Projection Postulate*) If an $\mathscr{A}$-measurement yields the result a_k, the immediate post-measurement state of the system measured is α_k, $A\alpha_k = a_k\alpha_k$.

This proposition, as stated, is absurd if only because it associates state vectors with single systems in the illicit, classical manner discussed in section **5**. Moreover, *P* cannot be accepted as a postulate because, as Margenau [18], [19] has shown, there exist realistic measurement procedures which in no sense satisfy *P*. Finally, from a practical viewpoint, Occam's Razor perhaps suffices to dismiss *P* from axiomatic status; for despite its frequent appearance in the opening chapters of quantum mechanics texts, *P* is never used in any calculation. Thus *P* is at once absurd, false, and useless.

There is, however, a statement resembling *P* which is at least not absurd, though its necessity and utility are certainly in doubt:

P′: (*Weak Projection Postulate*) If $\mathscr{A}$-measurements are performed on an ensemble, the post-measurement subensemble consisting of those systems which yielded a_k has density operator P_{α_k}.

P′, milder than P, only suggests that when the interaction ceases, a certain selection of subensembles would always be possible. Together with the other axioms, it does imply the measurement transformation $P_{\sum_k \langle \alpha_k, \psi \rangle \alpha_k} \to \sum_k |\langle \alpha_k, \psi \rangle|^2 P_{\alpha_k}$, but the fact remains that we have found no reason to adopt either that transformation or P′ as a *necessary* property of measurement. Indeed since the foregoing dissection of the Copenhagen interpretation has revealed its foundation to be a set of over-extended analogies to a highly subjective version of classical statistics, no supplementation of the quantum axioms even by P′ seems at all justifiable.

7. Outline of standard measurement theory. To provide a skeletal basis for subsequent discussions of its many ramifications, the "standard" quantum theory of measurement will now be outlined in an abstract mathematical fashion temporarily avoiding all philosophical problems of interpretation. Any actual measurement of an observable $\mathscr{A}$ on a quantum system $\mathscr{S}$ assumed to be an element of a known ensemble, is performed with an auxiliary system $\mathscr{M}(\mathscr{A})$ is called a measuring apparatus for observable $\mathscr{A}$, or $\mathscr{A}$-meter. This means that $\mathscr{S}$ and $\mathscr{M}$ physically interact so that known correlations arise between the possible measurement results of observable $\mathscr{A}$ and some observable $\mathscr{C}$ belonging to $\mathscr{M}$. Since $\mathscr{M}$ is an $\mathscr{A}$-meter, these correlations are sufficient to render a "direct" $\mathscr{A}$-measurement superfluous. Thus $\mathscr{A}$ is measured by "reading the $\mathscr{A}$-meter," i.e. by measuring $\mathscr{C}$ on $\mathscr{M}$.

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be the Hilbert spaces associated with $\mathscr{S}$ and $\mathscr{M}$, respectively. The tensor product space $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ is then appropriate for the study of the $\mathscr{S}$–$\mathscr{M}$ interaction. As usual, the operators corresponding to $\mathscr{A}$ and $\mathscr{C}$ will be denoted by $A \otimes 1$ and $1 \otimes C$. To avoid burdensome notation, we assume for the present that A and C have discrete, nondegenerate spectra. A, C satisfy $A\alpha_k = a_k\alpha_k$, $C\gamma_l = c_l\gamma_l$; $\{\alpha_k\}$ spans $\mathscr{H}_1$, $\{\gamma_l\}$ spans $\mathscr{H}_2$, and $\{\alpha_k \otimes \gamma_l\}$ spans $\mathscr{H}$.

Nothing significant comes from considering mixtures as opposed to pure states; we assume therefore that initially $\mathscr{S}$ and $\mathscr{M}$ are "in" pure states ψ and χ. (Having made the point that the ensemble must not be forgotten, we shall henceforth often use this common expression.) It can be shown that the composite system $\mathscr{S} + \mathscr{M}$ will then be in state $\psi \otimes \chi$.

By Th9 the temporal evolution of the state vector is always expressible by a linear evolution operator T: $\psi(t_2) = T(t_2, t_1)\psi(t_1)$. In the product space $\mathscr{H}$, if $\mathscr{S}$ and $\mathscr{M}$ do not interact, T is decomposable to $T_1 \otimes T_2$; conversely, an indecomposable T expresses interaction.

Now, according to the general principles stated above, the measurement process entails $\mathscr{S}$–$\mathscr{M}$ interaction leading to the establishment of correlations. Mathematically, this will be expressed as a condition on T_A, the indecomposable evolution operator for $\mathscr{A}$-measurement. That condition (correlation assumption) is almost always given as

$$T_A(\alpha_k \otimes \chi) = \alpha_k \otimes \gamma_k,$$

from which it follows that

$$T_A(\psi \otimes \chi) = T_A\Big(\sum_k \langle\alpha_k, \psi\rangle\alpha_k \otimes \chi\Big) = \sum_k \langle\alpha_k, \psi\rangle\alpha_k \otimes \gamma_k.$$

The desired correlation arises as follows: it can be shown from the axioms that the final composite state vector $\sum_k \langle\alpha_k, \psi\rangle\alpha_k \otimes \gamma_k$ means that, if an $\mathscr{A}$-measurement is performed on $\mathscr{S}$ and a $\mathscr{C}$-measurement on $\mathscr{M}$, the probability that the pair (a_k, c_l) will result is just $|\langle\alpha_k, \psi\rangle|^2\, \delta_{kl}$. Hence, the $\mathscr{C}$-measurement alone suffices.

Finally, it is customary to consider the post-measurement $\mathscr{S}$-ensemble independently; this is done by focusing on the measurement statistics for $\mathscr{S}$-observables only, i.e. those corresponding to operators of the form $B \otimes 1$. A simple calculation shows that the density operator $\hat{\rho}_1$ of that ensemble is given by[15]

$$\hat{\rho}_1 = \mathrm{Tr}_2 \, P_{\sum_k \langle \alpha_k, \psi \rangle \alpha_k \otimes \gamma_k} = \sum_k |\langle \alpha_k, \psi \rangle|^2 \, P_{\alpha_k}.$$

Thus the density operator for $\mathscr{S}$ has, in the course of the measurement act, changed from $P_{\sum_k \langle \alpha_k, \psi \rangle \alpha_k}$ to $\sum_k |\langle \alpha_k, \psi \rangle|^2 \, P_{\alpha_k}$, which is just the measurement transformation deemed so essential by the Copenhagen school.

8. An untenable consequence of the standard theory. In its crude form based on assignment of state vectors to single systems, the projection postulate (P) is easily used to "prove" that simultaneous measurement of noncommuting observables is impossible. All that is required is the observation that the post-measurement state, by virtue of wave packet reduction, would have to be simultaneously an eigenvector of two different operators. Unless the latter commute, such an eigenvector is a rarity; in fact usually none exists at all. Now, we hold that any proposition which declares impossible the simultaneous measurement of two observables necessarily stems from a false hypothesis. This follows from the fact that it is possible to construct within the quantal framework given by P1–P3 legitimate models[16] of simultaneous measurement schemes for noncommuting observables. Thus from a physical point of view, to say that the statement "$\mathscr{A}$ and $\mathscr{B}$ cannot be measured simultaneously" is an analytic truth simply condemns the axiom set from which it was derived. We have then essentially a *reductio ad absurdum* argument against the fanciful version of wave packet reduction, which has, however, already been rejected above on other grounds.

This raises the question as to whether the weaker (but strongest admissible) form of the projection postulate (P′) is also subject to such a critique; the answer is that P, as an isolated postulate, is not directly assailable along these lines because of its careful association of eigenvectors with ensembles rather than individual systems. Nevertheless, it turns out that P′, like P, does become untenable when confronted with basic quantal theorems; for like naive wave packet reduction, *it implies that two observables $\mathscr{A}$ and $\mathscr{B}$ are simultaneously measurable only if* $[A, B] = 0$.

To prove this, recall first that according to von Neumann's measurement transformation, an $\mathscr{A}$-measurement on a pure ensemble converts the density operator from P_ψ to the mixture $\sum_k |\langle \alpha_k, \psi \rangle|^2 \, P_{\alpha_k}$. Similarly, a $\mathscr{B}$-measurement would induce the change $P_\psi \to \sum_l |\langle \beta_l, \psi \rangle|^2 \, P\beta_l$. If this transformation is a *universal* property of measurement, a simultaneous measure of $\mathscr{A}$ and $\mathscr{B}$ must therefore be described by

$$P_\psi \to \hat{\rho} = \sum_k |\langle \alpha_k, \psi \rangle|^2 \, P_{\alpha_k} = \sum_l |\langle \beta_l, \psi \rangle|^2 \, P_{\beta_l}.$$

[15] Tr_2 signifies a trace operation involving only matrix indices representing $\mathscr{H}_2$.

[16] Such counterexamples to the widely believed principle of incompatibility of noncommuting observables are discussed in detail in [21].

According to the probability rules of quantum theory,

$$W_{\mathscr{B}}(b_l;\hat{\rho}) = \mathrm{Tr}(\hat{\rho}P_{\beta_l}),$$

which is independent of the particular representation of $\hat{\rho}$. Hence,

$$W_{\mathscr{B}}(b_l;\hat{\rho}) = \mathrm{Tr}(\hat{\rho}P_{\beta_l}) = \sum_k |\langle\alpha_k,\psi\rangle|^2\,\mathrm{Tr}(P_{\alpha_k}P_{\beta_l}) = \sum_k |\langle\alpha_k,\psi\rangle|^2\,|\langle\beta_l,\alpha_k\rangle|^2,$$

but also

$$W_{\mathscr{B}}(b_l;\hat{\rho}) = \mathrm{Tr}(\hat{\rho}P_{\beta_l}) = |\langle\beta_l,\psi\rangle|^2;$$

thus we have

$$|\langle\beta_l,\psi\rangle|^2 = \Big|\sum_k \langle\beta_l,\alpha_k\rangle\langle\alpha_k,\psi\rangle\Big|^2 = \sum_k |\langle\alpha_k,\psi\rangle|^2\,|\langle\beta_l,\alpha_k\rangle|^2.$$

Clearly, this does not hold for every ψ, $\{\alpha_k\}$, and $\{\beta_l\}$. To find conditions under which it is correct, note that

$$\begin{aligned}\Big|\sum_k \langle\beta_l,\alpha_k\rangle\langle\alpha_k,\psi\rangle\Big|^2 &= \sum_k \big(\langle\beta_l,\alpha_k\rangle\langle\alpha_k,\psi\rangle\big)^*\Big(\sum_n \langle\beta_l,\alpha_n\rangle\langle\alpha_n,\psi\rangle\Big)\\ &= \sum_k |\langle\alpha_k,\psi\rangle|^2\,|\langle\beta_l,\alpha_k\rangle|^2\\ &\quad + \sum_{n,k,n\neq k} \langle\alpha_k,\beta_l\rangle\langle\beta_l,\alpha_n\rangle\langle\psi,\alpha_k\rangle\langle\alpha_n,\psi\rangle.\end{aligned}$$

The measurement transformation is therefore applicable only to simultaneous $\mathscr{A}$, $\mathscr{B}$ measurements such that

$$\sum_{n\neq k} \langle\alpha_k,\beta_l\rangle\langle\beta_l,\alpha_n\rangle\langle\psi,\alpha_k\rangle\langle\alpha_n,\psi\rangle = 0, \tag{1}$$

which means that the measurement transformation can describe simultaneous measurements only for *some* pairs of observables, viz. those whose eigenvectors satisfy equation (1) for all ψ.

To find the relation between the sets of eigenvectors $\{\alpha_k\}$ and $\{\beta_l\}$, we consider special ψ's:

Let $\psi = (1/\sqrt{2})(\alpha_N + \alpha_K)$, $N \neq K$. Substitution into (1) gives

$$\begin{aligned}&\sum_{n\neq k} \langle\alpha_k,\beta_l\rangle\langle\beta_l,\alpha_n\rangle(\tfrac{1}{2})(\delta_{kN}+\delta_{kK})(\delta_{nN}+\delta_{nK})\\ &\qquad = \tfrac{1}{2}(\langle\alpha_N,\beta_l\rangle\langle\beta_l,\alpha_K\rangle + \langle\alpha_K,\beta_l\rangle\langle\beta_l,\alpha_N\rangle\\ &\qquad = \mathrm{Re}(\langle\alpha_N,\beta_l\rangle\langle\beta_l,\alpha_K\rangle) = 0.\end{aligned}$$

Similarly, let $\psi = (1/\sqrt{2})(\alpha_N + i\alpha_K)$, $N \neq K$, to get

$$\begin{aligned}&\sum_{n\neq k} \langle\alpha_k,\beta_l\rangle\langle\beta_l,\alpha_n\rangle(\tfrac{1}{2})(\delta_{kN}-i\delta_{kK})(\delta_{nN}+i\delta_{nK})\\ &\qquad = \frac{i}{2}(\langle\alpha_N,\beta_l\rangle\langle\beta_l,\alpha_K\rangle - \langle\alpha_k,\beta_l\rangle\langle\beta_l,\alpha_N\rangle)\\ &\qquad = -\mathrm{Im}(\langle\alpha_N,\beta_l\rangle\langle\beta_l,\alpha_K\rangle) = 0.\end{aligned}$$

Hence, equation (1) may be replaced by the simpler restriction

(2) $\langle \alpha_n, \beta_l \rangle \langle \beta_l, \alpha_m \rangle = 0$, for every $l, n, m, n \neq m$.

Since $\{\alpha_k\}$ and $\{\beta_l\}$ are *eigenvector* sets, neither contains the null vector. Thus, any α_M has the property that $\langle \beta_l, \alpha_M \rangle \neq 0$ for *some* value of l, say $l = L$. Equation (2) then implies that $\langle \alpha_n, \beta_L \rangle = 0, n \neq M$, i.e. β_L is orthogonal to every element of $\{\alpha_k\}$ except one, α_M. But $\{\alpha_k\}$ is a complete set; therefore α_M and β_L must be equal up to a phase factor (i.e. belong to the same ray in Hilbert space). The same argument applies to all values of L. Hence, each element of complete set $\{\beta_l\}$ is an element of complete set $\{\alpha_k\}$; the eigenvector sets for observables $\mathscr{A}$ and $\mathscr{B}$ are identical (except for unimportant phases). It follows that the operators A, B corresponding to simultaneously measurable observables $\mathscr{A}, \mathscr{B}$ *must commute*!

We have therefore proved that the measurement transformation

$$P_\psi \rightarrow \sum_k |\langle \alpha_k, \psi \rangle|^2 P_{\alpha_k}$$

forbids the simultaneous measurement of noncommuting observables.

Yet it is just this transformation which is widely accepted as a universal characteristic of measurement. Most presentations of the quantum theory of measurement, including von Neumann's [25] and London and Bauer's [16] classic treatments (the standard theory outlined in section 7), adopt it as a goal. A derivation of it typically counts not only as a general explanation of measurement but also as a demonstration of the internal consistency of quantum theory. We now see that this point of view must be rejected, for the transformation entails an absurd corollary, viz. that simultaneous measurements of noncommuting observables are impossible. Hence the transformation $P_\psi \rightarrow \sum_k |\langle \alpha_k, \psi \rangle|^2 P_{\alpha_k}$ cannot be upheld as a *defining* attribute of the quantum measurement process.

9. The apparatus as a "classical" system. When Schrödinger [23] offered his "*burleske*" quantum description of a cat in a box, he illustrated a point which many quantum theorists have taken seriously in connection with measurement theory. Schrödinger's cat is incarcerated in a chamber containing a few radioactive atoms and some equipment. The only interaction between cat and atoms occurs when an atom disintegrates, but that rare event will trigger some lethal machinery. A geiger counter responds to the decay by setting into operation a hammer which shatters a flask of cyanide. Thus the interaction correlates the possible states of the atoms with the "alive" and "dead" states of the cat.

Consider now a quantum theoretical description of the composite system consisting initially of the cat and one unstable atom. The cat-observable of interest is the proposition, "it is alive," with two eigenvalues, 1 (yes) and 0 (no), to which belong eigenstates α and δ, respectively. For the atom, let φ denote its initial unstable state and θ its possible stable state. At first, the composite system is in the state $\psi(0) = \alpha \otimes \varphi$; as time progresses, ψ develops into $\psi(t) = c_1(t)\alpha \otimes \varphi + c_2(t)\delta \otimes \theta$, which indicates the correlation between the two systems.

Reluctance to accept this as an adequate description of what has happened in the box stems from the unfortunate *literal* interpretation of the phrase "a system in state ψ." Thus it is said [15] that in actuality the state of a cat is never a blurred superposition of "living" and "dead" eigenstates, but is at all times one or the other, though which one might be unknown. To express this *ignorance* (recall the Copenhagen interpretation), supposedly a mixture is required. Even though we contend that this demand arises from a misinterpretation of ψ, nevertheless it seems at first that this mysterious desire for a mixture to describe the cat is automatically fulfilled anyhow. Indeed, the mortality statistics for the cat alone are easily calculated and the density operator is, as a matter of fact, mixed:

$$\rho_{\text{cat}}^{(t)} = |c_1(t)|^2 P_\alpha + |c_2(t)|^2 P_\delta.$$

Strangely enough, this does not satisfy the objectors; their demands are even stronger. Supposedly, it is an a priori truth that a cat-atom system *should* be described by the mixed, correlated density operator,

$$\rho(t) = |c_1(t)|^2 P_{\alpha\otimes\varphi} + |c_2(t)|^2 P_{\delta\otimes\theta},$$

which would refer of course to an imaginary ensemble representing our ignorance as to which of the two possibilities actually obtained. Unfortunately, the temporal evolution,

$$P_{\alpha\otimes\varphi} \to |c_1(t)|^2 P_{\alpha\otimes\varphi} + |c_2(t)|^2 P_{\delta\otimes\theta}$$

is absolutely impossible within the dynamical scheme of quantum theory, unless the composite system interacts with another system. However, it is pointless to multiply the number of systems, for at each state the same objections would arise, together with the same demand that an *impossible* total density operator is the "correct" one. Nor would it help to assume, as in Heisenberg's theory of measurement, that the cat-atom system, to be observed at all, must be immersed in an environment (Heisenberg's "external world") described by a mixture, say $\rho = \sum_k w_k P_{\gamma_k}$. It is true that the new total system—cat, atom, and surroundings—would then be in a mixed state at time t; but note closely its form (immediately derivable from the linearity of quantum dynamics):

$$\begin{aligned} P_{\alpha\otimes\varphi} \otimes \sum_k w_k P_{\gamma_k} &= \sum_k w_k P_{\alpha\otimes\varphi\otimes\gamma_k} \\ &\to \sum_k w_k P_{[c_1^{(k)}\alpha\otimes\varphi\otimes(\Sigma_n d_n^{(k)}\gamma_n) + c_2^{(k)}\delta\otimes\theta\otimes(\Sigma_m g_m^{(k)}\gamma_m)]}. \end{aligned}$$

Every component of the resultant mixture has a "blurred" cat in it!

Schrödinger's cat is of course a metaphor; what it represents is the notion of classical system, about which there are naturally many preconceptions. Chief among these is the cherished belief that a classical system cannot take part in statistical considerations which include the so-called "interference" of probabilities which occurs for quantum states. A classical system always possesses a definite value for every classical observable, although there may be ignorance as to *which* value; but if so, the associated probabilities do not "interfere." One might ask: why so much interest in prequantum ideas? After all, there is no such thing as a classical system, except in a special limiting case of quantum theory. Besides, as already suggested, the cat paradox is based on the unwarranted association of ψ with a *single* cat-atom system in an almost occult sense. Thus a superposition of two eigenstates for a classical system is regarded as a kind of unreal, smeared representation which does not recognize that at all times such systems *possess* either one eigenvalue or the other. We have already granted in section **4** that in the quantum framework observables are not possessed; but it is a gross distortion to say that a superposition of eigenstates represents anything "blurred". Consider again the two density operators for the cat-atom system:

$$(1) \qquad \rho = P_\psi, \qquad \psi = c_1\alpha \otimes \varphi + c_2\delta \otimes \theta$$

$$(2) \qquad \rho = |c_1|^2 P_{\alpha\otimes\varphi} + |c_2|^2 P_{\delta\otimes\theta}.$$

The physical meaning of (1) is just this: in an ensemble of cat-atom systems examined at time t, the fraction $|c_1(t)|^2$ of the systems will display a live cat (and unchanged atom); the fraction $|c_2(t)|^2$ will exhibit a dead one (and radioactive decay products). *Moreover,* (2) *means exactly the same thing* so far as the observables in question are concerned. As has already been discussed at length, it is improper to regard pure states as referring to single systems and mixed states to imaginary ensembles expressing ignorance. Every density operator, pure or mixed, has the same reference—an ensemble.

This is not to say that (1) and (2) are identical; in principle, there exist observables whose measurement statistics are different in the two cases; hence the only scientific way to show that (2) is preferable to (1) would be to study empirical measurement results for such an observable. In the absence of such evidence, there is no reason to prefer (2) to (1) provided quantum theory is understood, not in the Copenhagen interpretation, but rather as outlined in sections **2–5**. To insist that a composite system which is partly "classical" cannot be in a superposition of eigenstates is therefore quite dogmatic.

Nevertheless, many theories of measurement differ from the standard one given in section **7** by imposing the additional requirement upon $\mathscr{M}$ that it be *classical.* As a result, much effort is expended in formulating reasons for replacing the inevitable post-measurement pure state of $\mathscr{S} + \mathscr{M}$ by a mixture. One method of justifying replacement of the pure state by a mixture is to define some sense in which the two are equivalent; the definition would also serve to identify the "classical" level within a quantal context. But in what sense can two unequal density operators, $\rho^{(1)}$ and $\rho^{(2)}$, be physically "equivalent"? Clearly they are distinguishable only by comparison of the measurement statistics they entail. Thus $\rho^{(1)}$ and $\rho^{(2)}$ are certainly equivalent if, for every A, $\mathrm{Tr}(\rho^{(1)}A) = \mathrm{Tr}(\rho^{(2)}A)$; in fact this is only true in the extreme case when $\rho^{(1)} = \rho^{(2)}$. If, on the other hand, only a restricted set of operators, $\mathsf{A} \equiv \{A_k\}$, is considered, $\rho^{(1)}$ and $\rho^{(2)}$ will be indistinguishable relative to A-measurements, provided $\mathrm{Tr}(\rho^{(1)}A_k) = \mathrm{Tr}(\rho^{(2)}A_k)$, for every A_k in the set A. This concept, which we shall call A-equivalence and denote by $\rho^{(1)} \overset{\mathsf{A}}{\sim} \rho^{(2)}$, is occasionally used to secure the desired post-measurement mixture.

In terms of the notation introduced in section **7**, the problem of measurement, for theorists worried about the "classical" aspect of $\mathscr{M}$, is now reduced to the following: find a meaningful restriction to place on A such that $\rho^{(1)} \overset{\mathsf{A}}{\sim} \rho^{(2)}$, where $\rho^{(1)} = P_{\sum_k \langle \alpha_k, \psi \rangle \alpha_k \otimes \gamma_k}$, $\rho^{(2)} = \sum_k |\langle \alpha_k, \psi \rangle|^2 P_{\alpha_k \otimes \gamma_k}$. The observables corresponding to operators in A are then called the "classical" ones, i.e. those directly apprehended by the laboratory physicist, who cannot therefore distinguish $\rho^{(1)}$ from $\rho^{(2)}$. Examples of quantum measurement theories in which A-equivalence plays this role are those of Feyerabend [8], Wakita [26], and Jauch [13]; but these are motivated more or less by an understanding of basic quantum theory in which Schrödinger's cat allegory is a *paradox*. We have already considered this position and dismissed it as an unfortunate byproduct of the Copenhagen interpretation.

In discourses on complementarity, Bohr repeatedly insisted that *classical* description plays a role in quantum theory which is unavoidable and of fundamental significance. His often quoted declaration [3], "... *however far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms*," has been echoed again and again. Heisenberg, for example, notes that the *language* of the laboratory employs the concepts of classical physics, and then asserts ([12], p. 44) that "we cannot and should not replace these concepts by any others." Such remarks go far beyond the milder and more reasonable asymptotic requirement that no quantal prediction concerning classically describable "macroscopic experience" should contradict valid classical prediction. Strictly, of course, there is always theoretical contradiction in the sense that quantal and classical constructs are quite different[17]; the correspondence principle can require only *empirical* agreement. This, however, is not the point stressed in the foregoing quotations.

Are we logically forced to accept the claim that classical physics is the cornerstone of quantum theory? Must the language of the laboratory be classical? To answer the first question, contrast the correspondence postulate (P1) as presented in section **2** with the following popular formulation which does make classical mechanics appear to be the basis of quantum mechanics:

> P1a: The observables q (position) and p (momentum) correspond to operators Q, P which satisfy $[Q, P] = i\hbar 1$. Any observable $\mathscr{A}$, whose state function in *classical* mechanics is $\mathscr{A}(q, p)$ corresponds to a Hermitean operator of the form $A = \mathscr{A}(Q, P)$.

Note that the very concept of observable is here construed to be basically classical; quantal representatives of observables are generated from their classical analogues. However, the effectiveness of this procedure (which, incidentally, is not always logically consistent) is obviously limited by the fact that quantum theory considers observables for which no classical analogue is imaginable. Nevertheless, quantum field theory, for example, is often introduced "heuristically" or "inductively" by generalizing P1a to an unconvincing method called "quantization." The concept of quantum field is then induced from a bizarre analysis of classical con-

[17] This point has been discussed at length by Bohm [2], Feyerabend [8], and Hanson [10].

tinuum mechanics in which field strengths become, upon "quantization," noncommuting field operators.

Actually, P1a and its generalizations are not required at all among the basic principles of quantum theory. The notion that classical physics is the foundation of quantum physics has an evident *historical* origin, but is of no logical value. Both theories have the same epistemological status as verified connections among their constructs, which are related in well defined ways to the given, the data of empirical experience. However, for historical reasons and because quantal and classical accounts must be empirically compatible within the classical sphere of interest, many quantal rules of correspondence appear to be based on classical physics. Bergmann [1] made much of this in his "logic of quanta." Nevertheless, this is essentially a backward-looking position; the classical world view, properly understood, is not self-evident, nor is it forced upon us by percepts. Like quantum theory, its logical genesis was an act of scientific creativity, or construction. Thus it seems preferable to formulate the correspondence postulate (P1) as in section **2**, a statement which recognizes no logical dependence of quantum theory upon classical physics. The "quantization" process (P1a) is then diminished to its correct status as a mnemonic device sometimes useful to classically trained physicists.

As for the second question, from the same philosophic perspective, it is clear that classical physics need not and perhaps ultimately should not be the standard mode of experiment description. The reasonable assertion that laboratory procedures be reported in communicable, "common-sense" language simply does not imply what Bohr and Heisenberg suggest, viz. that whatever experimental operations are performed must be described classically. Consider, for example, the complex of sensations which we categorize as the "motion of a Maxwell top" (an antique device seemingly as "classical" as anything could ever be). The primitive datal percepts involved are certainly *neither classical nor quantal*; moreover, these terms are not necessarily applicable to the empirical constructs used to describe and quantify observations and results of operations on the top. Only the far more abstract constructs and their interconnections which comprise the physical theory created to explain these empirical observations can be reasonably called classical or quantal. However, when a given theory is well entrenched, this "epistemological depth" of its constructs is forgotten in practice, and experiments come to be described in abstract terms provided by the theory itself. In the case of Maxwell's top, an empirical fact of interest is the variation in wobbling patterns which accompanies adjustments of the screws on the sides of the top; but a complete report of this observation is communicable without the sophisticated concepts of any physical theory, although such a description would be cumbersome and verbose indeed. But since classical mechanics provides the established theory of the top, the changing patterns of its motion occasioned by screw adjustments may well be described in terms of "observations of the dependence of angular velocity upon the inertia tensor"—a truly "classical" laboratory language. Hence familiarity with a successful theory (classical mechanics) has created the illusion that its profound constructs are directly perceptible or self-evident; i.e. the "classical" laboratory language comes to be regarded as necessary, an unfortunate epistemological mistake. After all, quantum theory, too, can fully explain the observed wobblings of the top; and it could even provide a "quantal" laboratory language, familiarity with which can, and perhaps some day will, lead to its adoption as the "necessary" vernacular of common sense description. We therefore reject the principle that the perceived world is somehow inherently "classical" and that the quantum theory of measurement must have a "classical" aspect.

One of the trends (the "A-equivalence theories") in this kind of measurement theory was mentioned earlier. The approach was rather formal, the method being to "define away" allegedly undesirable interference terms. Another way to secure the desired "classical" aspect apparently originated with Jordan [14], who advocated thermodynamic analysis of the measuring apparatus. However, again the underlying purpose is apparently to derive the von Neumann measurement transformation, which is, as we have seen before, the recurrent goal of most measurement theories. A recent elaborate attempt along these lines due to Daneri, Prosperi, and Loinger [7] has been endorsed by Rosenfeld [22], an outspoken apologist for Copenhagen ideas (Bohr's in particular). In their theory, the measurement transformation is derived by expressing the "classical" nature of apparatus in terms of ergodicity conditions and defining macro-observables to be temporal averages of quantal observables. It is then shown that the quantal dynamics of $\mathscr{S} + \mathscr{M}$, if supplemented by these conditions, effectively yields the measurement transformation and explains the registration of a permanent "reading" in $\mathscr{M}$. According to Rosenfeld, "The main purpose of the analysis of measurement is to exhibit the physical process to which this formal 'reduction' [the measurement transformation] corresponds," and this Daneri–Prosperi–Loinger theory fulfills that requirement.

Probably such a demonstration does offer an approximate explanation of some actual measurement schemes; but as we have already observed, its basic structure is the derivation of an unnecessary, even rare, property (the measurement transformation) from an erroneous metaphysical belief (that apparatus is inherently "classical").[18]

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