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Thermodynamic Aspects of Schrödinger's Probability Relations

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Using Schrödinger's generalized probability relations of quantum mechanics, it is possible to generate a canonical ensemble, the ensemble normally associated with thermodynamic equilibrium, by at least two methods, statistical mixing and subensemble selection, that do not involve thermodynamic equilibration. Thus the question arises as to whether an observer making measurements upon systems from a canonical ensemble can determine whether the systems were prepared by mixing, equilibration, or selection. Investigation of this issue exposes antinomies in quantum statistical thermodynamics. It is conjectured that resolution of these paradoxes may involve a new law of motion in quantum dynamics.

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1. CANONICAL STATES: PREPARATION AND VERIFICATION

Though Schrödinger has long been enshrined in our textbooks, and quite properly, as one of the creators of quantum mechanics, it is noteworthy that he was in fact also one of its severest critics. Fifty years ago, shortly after the appearance of the now famous protest of Einstein, Podolsky, and Rosen, Schrödinger demonstrated that they had exposed only the tip of the iceberg. In a brilliant pair of papers(1,2) concerning "probability relations between separated systems," he revealed startling generalizations which remain to this day largely unknown. By contrast, the special EPR case, after subsequent quantification in the Bell inequalities, finally became experimentally interesting and is today deservedly notorious.

Schrödinger, like Einstein, deplored certain logical consequences of quantum mechanics. In particular, he characterized his generalized probability relations as "conclusions unavoidable within the present theory but repugnant to some physicists including the author." The present paper,

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written as we celebrate the Schrödinger Centenary, will focus more on the unavoidability than the repugnance as we explore the special consequences of Schrödinger's relations when they are extended specifically to quantum thermodynamics.

To set the stage, let us imagine that an experimenter X^j has been provided with a reproducible procedure that prepares a physical system $\mathscr{H}^j$ for study. In addition, X^j has at his disposal measurement devices that give operational meaning to all the observables A^j, B^j,... of $\mathscr{H}^j$. If X^j takes as his mission the determination of the quantum state ρ^j that describes systems prepared in the manner prescribed, he must in general execute the preparatory procedure many times, generating thereby a statistical ensemble, and in each run obtain a datum for some observable by employing the appropriate device. In principle, he eventually will have gathered enough data on, say, observable A^j to compute the mean value $\langle A^j \rangle$. Continued repetition would finally yield a set of statistics $\{\langle A^j \rangle, \langle B^j \rangle, \ldots\}$ rich enough to determine the state ρ^j. The list of observables $\{A^j, B^j, \ldots\}$ whose means are sufficient to find ρ^j has been called a quorum.(3)

For simplicity we adopt a mathematical notation in which the same symbol will be used both for a physical concept and its mathematical representative in quantum mechanics. Thus $\mathscr{H}^j$ denotes a Hilbert space; A^j, B^j,..., self-adjoint operators; and ρ^j, the density operator. The quantal mean value $\langle A^j \rangle$ is then given by

$$\langle A^j \rangle = \mathrm{Tr}_j(\rho^j A^j) \tag{1}$$

where Tr_j denotes the trace over Hilbert space $\mathscr{H}^j$. The various symbols are superscripted in anticipation of later considerations of multicomponent systems where, for example, $\mathscr{H}^1 \otimes \mathscr{H}^2$ will denote both the physical composite of systems $\mathscr{H}^1$ and $\mathscr{H}^2$ and the tensor product space that represents it mathematically. In this case ρ^{12} is the total density operator for $\mathscr{H}^1 \otimes \mathscr{H}^2$, and ρ^1, for example, is the reduced density operator

$$\rho^1 = \mathrm{Tr}_2 \rho^{12} \tag{2}$$

Consider for the moment only system $\mathscr{H}^1$ and its observer X^1. Suppose that X^1 has gathered sufficient data to compute the means for a quorum of observables of $\mathscr{H}^1$ and from them has concluded that ρ^1 has the form

$$\rho^1 = \hat{\rho}^1(\beta^1) \tag{3}$$

where $\hat{\rho}^1(\beta^1)$ is the well-known canonical density operator defined by

$$\hat{\rho}(\beta) = \frac{e^{-\beta H}}{Z(\beta)} \tag{4}$$

with

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H} \tag{5}$$

H being the energy operator of the system. Obviously the preparation scheme generating the statistical ensemble from which X^1 has determined ρ^1 is very specialized, for the canonical state $\hat{\rho}^1(\beta^1)$ is far from arbitrary. Indeed such canonical states are subject in practice to a great deal more interpretation and application than are other density operators because of their unique role in thermodynamics. At this point X^1, having determined (3), can contribute nothing more to the study of $\mathscr{H}^1$, since (3) and the mean value formula (1) now enable prediction of all further statistical data that X^1 could obtain by measurement of the remaining observables of $\mathscr{H}^1$ (those outside the quorum).

Nevertheless, particularly in view of the familiarity of (3), it is difficult to restrain speculation as to the nature of the behind-the-scenes machinations that might have provided X^1 with an ensemble described by (3). Interestingly, there are at least three conceptually distinct methods that could have been invoked to generate the ensemble that was presented to X^1: (a) statistical mixing, (b) thermal equilibration, and (c) subensemble selection through correlations. In the sections that follow we shall illustrate each of these possibilities in detail.

As we proceed it should be kept in mind that, according to orthodox quantum mechanics, X^1 cannot distinguish among the three methods even though he has the empirical means to measure *every* quantal observable of $\mathscr{H}^1$. His density operator ρ^1 completely summarizes all information of the kind he can measure; yet each of the three methods we shall discuss would provide X^1 with the *same* state (3). Like any quantum mechanician, X^1 only knows that which he can verify—the mean values of all the observables of his system $\mathscr{H}^1$. Thus for X^1 the distinctions we shall explore are seemingly in a black box beyond his reach.

Whether these illustrations of distinct canonical state preparation methods are taken as sublime mysteries of nature or as evidences of a repugnant theoretical incompleteness, they surely disclose some intriguing features of contemporary quantum theory.

2. AMBIGUITY OF MIXTURES

The canonical state (3) is of course a mixture, and as such the ensemble presented to X^1 could have been manufactured simply by mixing pure states in the manner suggested by the spectral expansion

$$\rho^1 = \hat{\rho}^1(\beta^1) = \frac{e^{-\beta^1 H^1}}{Z(\beta^1)} = \sum_j \hat{p}_j |\psi_j^1\rangle\langle\psi_j^1| \tag{6}$$

where

$$H^1\psi_j^1 = E_j^1\psi_j^1, \quad \text{and} \quad \hat{p}_j = e^{-\beta^1 E_j^1}/Z(\beta^1)$$

(The ψ_j^1 and all other vectors appearing below are assumed normalized to unity.)

This scheme may be described metaphorically as follows: whenever X^1 requests an $\mathscr{H}^1$ from the black box, the preparing demon within always ejects to X^1 a system prepared in one of the eigenstates ψ_j^1; *which* one is selected randomly for each run but relative frequency overall for ψ_j^1 is maintained at $\hat{p}_j$. Such a construction of the canonical state is compatible with the common view in statistical thermodynamics that mixtures are merely representations of human ignorance. Thus each system $\mathscr{H}^1$ that X^1 inspects is "actually" in some pure state ψ_j^1; but since X^1 lacks such detailed knowledge, he expresses his ignorance by assigning $\mathscr{H}^1$ the mixed state (6).

There is a curious sense in which this ignorance interpretation of mixtures is flawed. Suppose the demon in the box has available instead of energy eigenstates $\{\psi_j^1\}$ a completely different, not necessarily orthogonal, set $\{\varphi_i^1\}$. If these are provided to X^1 according to a certain probability distribution $\{w_i\}$, X^1 will conclude from his quorum measurements that

$$\rho^1 = \sum_i w_i |\varphi_i^1\rangle\langle\varphi_i^1| \tag{7}$$

But suppose further that $\{w_i\}$ and $\{\varphi_i^1\}$ are carefully chosen so that ρ^1 in (7) is identical to the canonical ρ^1 in (6); i.e.,

$$\rho^1 = \sum_j \hat{p}_j |\psi_j^1\rangle\langle\psi_j^1| = \sum_i w_i |\varphi_i^1\rangle\langle\varphi_i^1| \tag{8}$$

For X^1 this is nothing more than two mathematical expressions for the same quantum state ρ^1, which in itself completely summarizes all statistics for all data sets X^1 can measure. In other words, for X^1 the distinct structures of the expansions themselves have no physical meaning and are in fact undiscoverable.

Against this background, the ignorance interpretation of the density operator as used in information-theoretic statistical thermodynamics seems shaky. If one really wants to express the idea that a system is definitely in some energy eigenstate but *which* one is unknown, then two lists must be given: the states $\{\psi_j^1\}$ and the corresponding subjective probabilities $\{\hat{p}_j\}$. Only then has it been recorded what one is ignorant of and the extent of that ignorance. Similarly, the specification $(\{\varphi_i^1\}, \{w_i\})$ would indicate that the system is in some unknown state from among the $\{\varphi_i^1\}$, the extent of ignorance as to which state being measured by the $\{w_i\}$. In the mathematics of quantum theory, multiple expansions like (8) are quite normal, and therefore ρ^1 itself simply does not capture the idea that subjective probabilities have been assigned to an underlying list of pure states.

The statistics of all the ordinary quantum observables X^1 is equipped to measure are all accounted for by ρ^1; if X^1 is to believe that the structure of the *expansion* of ρ^1 is meaningful, that there is something more to be known than ρ^1, he must be provided with additional observables whose measured values are functions not just of ρ^1 as a whole but of lists like $(\{\varphi_i^1\}, \{w_i\})$. We shall return to this point in the next section.

To demonstrate that the ambiguity suggested by (8) actually obtains for the canonical state, we shall apply a procedure adapted from Schrödinger's general analysis of mixtures. What Schrödinger proved was that any mixture ρ can be regarded as containing as one of its constituents a pure state φ that may be chosen almost *arbitrarily*, the only restriction on φ being that it must be orthogonal to the eigenspace of ρ belonging to eigenvalue zero.

If X^1 applies Schrödinger's procedure to the analysis of (6), the following steps are involved:

(1) Arbitrarily choose a pure state φ_1^1 that is to be considered a constituent of the mixture ρ^1. Note that unless $\beta^1 \to \infty$ (absolute zero), the canonical ρ^1 has no zero eigenvalue; hence φ_1^1 is *completely arbitrary.*

(2) Define the unit vector

$$\alpha_1^1 = \sum_j \frac{\langle \psi_j^1 | \varphi_1^1 \rangle}{\sqrt{\hat{p}_j}} \left(\sum_l \frac{|\langle \psi_l^1 | \varphi_1^1 \rangle|^2}{\hat{p}_l} \right)^{-1/2} \psi_j^1 \tag{9}$$

(3) Beginning with α_1^1, construct a complete orthonormal set of vectors spanning $\mathscr{H}^1$: $\{\alpha_1^1, \ldots, \alpha_i^1, \ldots\}$.

(4) Define the unit vectors

$$\varphi_i^1 = \sum_j \sqrt{\hat{p}_j} \, \langle \psi_j^1 | \alpha_i^1 \rangle \left(\sum_l \hat{p}_l |\langle \psi_l^1 | \alpha_i^1 \rangle|^2 \right)^{-1/2} \psi_j^1 \tag{10}$$

Note that φ_1^1 is the vector chosen arbitrarily in step (1).

(5) Canonical state ρ^1 may now be expressed as

$$\rho^1 = \sum_j w_i |\varphi_i^1\rangle \langle \varphi_i^1| \tag{11}$$

where

$$\begin{aligned} w_i &= (\langle \varphi_i^1 | \rho^{-1} | \varphi_i^1 \rangle)^{-1} \\ &= \left(\sum_l \frac{|\langle \psi_l | \varphi_i \rangle|^2}{\hat{p}_l} \right)^{-1} \end{aligned} \tag{12}$$

Thus, based on his measurements that determined ρ^1 and the above steps of mathematical analysis, X^1 can just as reasonably believe that the canonical ensemble was prepared by a demon using the prescrition (11) as by one using the conventional spectral expansion in (6). Given the total arbitrariness of φ_1^1, a parsimonious X^1 would surely conclude, as emphasized earlier, that only ρ^1 itself—not its expansion—is physically meaningful.

Nevertheless, it remains true that the ensemble provided to X^1 and from which he deduced ρ^1 could have been manufactured by any one of an infinite variety of statistical mixing schemes, most of which involve as raw material states $\{\varphi_i^1\}$ that are *not* eigenstates of energy. We face therefore a strange antimony: the canonical ensemble, so commonly and successfully employed to study systems in thermodynamic equilibrium, may itself be constructed from nonstationary constituents! This suggests that perhaps there is more to the concept of thermodynamic equilibrium than can be captured in the canonical density operator itself.

3. EQUILIBRATION AND AVAILABILITY

From the standpoint of thermodynamics, the canonical state is particularly significant because it is the unique solution to two constrained extremum problems. If one specifies the mean energy

$$\mathscr{U}(\rho) = \mathrm{Tr}\,(\rho H) \tag{13}$$

then the unique ρ which maximizes entropy

$$\mathscr{S}(\rho) = -\mathrm{Tr}\ \rho ln\rho \tag{14}$$

is the canonical $\hat{\rho}(\beta)$, where β is determined by (13). On the other hand, if one specifies $\mathscr{S}$, then the unique ρ which minimizes $\mathscr{U}$ is the canonical $\rho(\beta)$, where β is determined by (14). These singular properties make $\rho(\beta)$ the only quantum state that can be associated with the thermodynamic notion of stable equilibrium with all the practical consequences that entails.

Equilibration, the process wherein a closed physical system relaxes to a state of maximum $\mathscr{S}$ compatible with its conserved value of $\mathscr{U}$, is a commonplace phenomenon, though the details are hardly understood at all. However, the fact of equilibration permits us to invent a second method for generating an ensemble to which X^1 would assign the canonical state (3). Returning to the metaphor of the black box, we imagine that the preparing demon within engages in no statistical mixing whatever. Instead he has in his cellar numerous replicas of $\mathscr{H}^1$, each having the mean energy

$\mathscr{U}^1$ associated with the desired value of β^1. Whenever X^1 requests an $\mathscr{H}^1$ from the black box, the demon ejects an $\mathscr{H}^1$ that has been properly aged, i.e., permitted to equilibrate.

Once again it is clear from a strictly quantum-mechanical perspective that X^1, despite being able to measure all quantal observables, cannot distinguish the systems $\mathscr{H}^1$ that originated from statistical mixing from those that emerged from equilibration since both methods produced the same ρ^1, and ρ^1 contains all quantal information. However, if one accepts thermodynamics as a fundamental physical science and not merely as a branch of statistical inference theory, then there are compelling reasons to believe that X^1 ought to be able to distinguish somehow the cases of mixing and equilibration.

The pioneer effort to make this distinction was made by Hatsopoulos and Gyftopoulos,(4) who introduced and attempted to formalize the notion of ensemble *ambiguity*. In this language, all of the statistical mixing schemes considered above would provide X^1 with an ambiguous state ρ^1 while the equilibration method would generate an unambiguous ρ^1.

It is apparent that if X^1 is to be able to discern ambiguity through measurements, he must be endowed with a larger universe of observables than that normally contemplated in quantum mechanics. One possibility, mentioned previously, would be observables whose measured values were functions of mixing structure $(\{\varphi_i^1\}, \{w_i\})$. Although such observables are not recognized at present in normal quantum theory, it is noteworthy that their existence is in effect tacitly acknowledged in quantum statistical thermodynamics. For instance, if we take it as meaningful to speak of measuring the entropy of a system $\mathscr{H}^1$, then the average entropy that would be measured by X^1 on an ensemble generated by statistically mixing subensembles characterized by density operators $\{\rho_j^1\}$ according to probability distribution $\{W_j\}$ would be

$$\bar{\mathscr{S}}(\{\rho_j^1\}, \{W_j\}) = \sum_j W_j \mathscr{S}(\rho_j^1) \tag{15}$$

Since $\bar{\mathscr{S}}$ depends on the structure $(\{\rho_j^1\}, \{W_j\})$ and not just on ρ^1, it exemplifies an extraquantal observable of the kind we need. It should be noted especially that if X^1 can, as we have assumed, measure $\mathscr{S}$ on each $\mathscr{H}^1$, then he can certainly measure $\bar{\mathscr{S}}$ for the ensemble *even though he is ignorant of the structure* $(\{\rho_j^1\}, \{W_j\})$; to do so, he need only measure $\mathscr{S}$ on each $\mathscr{H}^1$ and then average his data.

Now, in the specific cases of statistical mixing considered earlier, the canonical ensemble was mixed either in the conventional manner $(\{\psi_j^1\}, \{\hat{p}_j\})$ or in one of the infinity of variations $(\{\varphi_i^1\}, \{w_i\})$ based on Schrödinger's procedure. Since $\mathscr{S}$ as defined in (14) vanishes for any pure state, it follows that the average entropy in any ensemble generated by such mixing of pure states must vanish:

$$\tilde{\mathscr{S}}(\{\psi_j^1\}, \{\hat{p}_j\}) = \tilde{\mathscr{S}}(\{\varphi_i^1\}, \{w_i\}) = 0 \tag{16}$$

In sharp contrast, the canonical ensemble generated by equilibration will have systems whose entropies are not only nonzero but in fact maximal for the given energy. Therefore, if X^1 can measure entropy in addition to the usual quantal observables, he can easily distinguish an ambiguous canonical ensemble from an unambiguous one.

To appreciate this point more fully and in a sense more empirically, it is fruitful to consider the concept of adiabatic availability as defined by Hatsopoulos and Gyftopoulos. Thermodynamics has historic roots in engineering problems about efficiency in the performance of work. If $\mathscr{H}^1$ does work on another system $\mathscr{H}^0$ (in an adiabatic process involving no other systems), then according to thermodynamics the maximum energy that $\mathscr{H}^1$ can transfer to $\mathscr{H}^0$ is the adiabatic availability of $\mathscr{H}^1$. Let us now assume that it is meaningful to speak of measuring the adiabatic availability of a system $\mathscr{H}^1$. The average adiabatic availability of an ensemble is then also well defined.

For a thermodynamic system $\mathscr{H}$ in (unambiguous) quantum state ρ, the adiabatic availability is given by

$$\mathscr{A}(\rho) = \mathscr{U}(\rho) - \mathscr{U}(\hat{\rho}(\beta)) \tag{17}$$

with β determined by

$$\mathscr{S}(\rho) = \mathscr{S}(\hat{\rho}(\beta))$$

The adiabatic availability of a system left in a canonical state after equilibration is zero; the adiabatic availability of a pure state ψ is

$$\mathscr{A}(|\psi\rangle\langle\psi|) = \langle\psi, H\psi\rangle - E_0 \tag{18}$$

where E_0 is the least eigenvalue of H.

Now, if X^1 is presented with a canonical ensemble generated by any of the statistical mixing schemes described above, he will find that the average adiabatic availability for the ensemble is

$$\begin{aligned}\tilde{\mathscr{A}}(\{\varphi_i^1\}, \{w_i\}) &= \sum_i w_i \mathscr{A}(|\varphi_i^1\rangle\langle\varphi_i^1|) \\ &= \sum_i w_i(\langle\varphi_i^1| H^1 |\varphi_i^1\rangle - E_0^1) \\ &= \mathscr{U}(\rho^1) - E_0^1 \end{aligned} \tag{19}$$

Since E_0^1 is the least eigenvalue of H^1, $\mathscr{A}$ is always nonnegative and is nonzero unless canonical state ρ^1 has $\beta \to \infty$ (absolute zero). We conclude that X^1 will be able to harness each system $\mathscr{H}^1$ so as to perform, on average, positive work on another system $\mathscr{H}^0$.

Suppose, on the other hand, that the canonical ensemble provided to X^1 has been generated not by mixing but rather by the equilibration method. In this case the adiabatic availability of each $\mathscr{H}^1$ is zero and hence the average availability is also zero. Therefore, if X^1 can indeed determine whether the systems $\mathscr{H}^1$ that he receives can or cannot, on average, do work, then he can indeed distinguish an ambiguous canonical ensemble from an unambiguous one.

Finally, there is another relation involving adiabatic availability that might serve as an analytical tool for an observer trying to ascertain the extent of ambiguity in an ensemble. In (19) we dealt with mixing cases where the constituents were all pure; interestingly, in this limit the average availability turned out to be a function of ρ^1 rather than structure $(\{\varphi_i^1\}, \{w_i\})$. However, for more general mixing $(\{\rho_j\}, \{W_j\})$, the average adiabatic availability has the form

$$\tilde{\mathscr{A}}(\{\rho_j\}, \{W_j\}) = \sum_j W_j[\mathscr{U}(\rho_j) - \mathscr{U}(\hat{\rho}_j(\beta_j))] \tag{20}$$

where β_j is determined by $\mathscr{S}(\rho_j) = \mathscr{S}(\hat{\rho}_j(\beta_j))$. Since $\mathscr{U}(\hat{\rho}_j(\beta_j)) \geqslant E_0$ (equality only when $\beta_j \to \infty$) it follows from (20) that

$$\tilde{\mathscr{A}}(\{\rho_j\}, \{W_j\}) \leqslant \mathscr{U}(\rho) - E_0 \tag{21}$$

where

$$\rho = \sum_j W_j \rho_j$$

Comparing (19) and the r.h.s. of (21), we conclude that the average adiabatic availability of a mixture of nonpure states $\{\rho_j\}$ is generally less than that of a mixture of pure states.

4. ARBITRARINESS OF CORRELATIONS

The literature of quantum mechanics abounds with examples of correlated systems that behave in perplexing, if not truly paradoxical, ways: system–apparatus relations in quantum measurement theory, EPR experiments both *gedanken* and actual, Schrödinger's cat, Wigner's friend, etc. All of these are special cases of Schrödinger's theory of probability relations between separated systems.

Using the latter we can design a scheme of amazing generality for delivering to X^1 a wide variety of ensembles, including the canonical one. The method involves a composite $\mathscr{H}^{123}=\mathscr{H}^1\otimes\mathscr{H}^2\otimes\mathscr{H}^3$ of three distinguishable systems, $\mathscr{H}^1$ being the same one discussed previously upon which X^1 measures quantal observables. The three systems need not be interacting; none of the results to be derived can be attributed to forces. The demons in this scenario are observers X^2 and X^3 who are equipped to measure all the quantum observables of $\mathscr{H}^2$ and $\mathscr{H}^3$ respectively.

We begin by writing an arbitrary state vector for $\mathscr{H}^1\otimes\mathscr{H}^2\otimes\mathscr{H}^3$ as

$$\Psi^{123}=\sum_{jkl} a'_{jkl}\alpha^1_j\alpha^2_k\alpha^3_l \tag{22}$$

where $\{\alpha^i_j \mid j=1,\dots,\infty\}$ is a complete orthonormal basis for $\mathscr{H}^i$. At first we are interested only in two systems, the composite $\mathscr{H}^{12}=\mathscr{H}^1\otimes\mathscr{H}^2$ and $\mathscr{H}^3$; it is therefore convenient to relabel $\{\alpha^1_j\alpha^2_k\}$ as $\{\alpha^{12}_m\}$ and a'_{jkl} as a'_{ml} so that

$$\Psi^{123}=\sum_{ml} a'_{ml}\alpha^{12}_m\alpha^3_l \tag{23}$$

The reduced density operator for $\mathscr{H}^3$ is defined by the partial trace

$$\rho^3=\mathrm{Tr}_{12}(|\Psi^{123}\rangle\langle\Psi^{123}|) \tag{24}$$

and is nonnegative definite for any Ψ^{123}. However, ρ^3 will be positive definite (all eigenvalues strictly positive) only for a subset of the vectors in $\mathscr{H}^{123}$. We shall refer to any state Ψ^{123} for which ρ^3 is positive definite as an *ovum state* for $\mathscr{H}^{12}$.

Henceforth let Ψ^{123} denote an *arbitrary ovum state* for $\mathscr{H}^{12}$. With Schrödinger's technique we can show how observer X^3 can, without interacting with $\mathscr{H}^{12}$, prepare $\mathscr{H}^{12}$ in *any* desired pure state Ψ^{12}. By using this astonishing power judiciously, X^3 can generate for X^1 the canonical ensemble, but the new method involves neither mixing nor equilibration. In order to achieve this goal, X^3 first contrives to prepare $\mathscr{H}^{12}$ in what we shall call the *desideratum state*

$$\hat{\Psi}^{12}=\sum_j \sqrt{\hat{p}_j}\,\psi^1_j\psi^2_j \tag{25}$$

where $\hat{p}_j$ and ψ^1_j are defined as in (6) and $\{\psi^2_j\}$ may be any orthonormal basis for $\mathscr{H}^2$. Letting $\{\varphi^3_n\}$ denote a complete set of orthonormal eigenvectors of ρ^3, we obtain the spectral expansion

$$\rho^3=\sum_n r_n|\varphi^3_n\rangle\langle\varphi^3_n| \tag{26}$$

where, because Ψ^{123} is an ovum state, $r_n>0$, for all n.

When α_l^3 is expanded using $\{\varphi_n^3\}$, (24) becomes

$$\Psi^{123} = \sum_{mn} a_{mn} \alpha_m^{12} \varphi_n^3 \tag{27}$$

where

$$a_{mn} = \sum_l a'_{ml} \langle \varphi_n^3 | \alpha_l^3 \rangle$$

From (24) and (27) it follows that

$$\rho^3 = \sum_{nn'} \left(\sum_l a_{ln} a_{ln'}^* \right) |\varphi_n^3\rangle \langle \varphi_{n'}^3| \tag{28}$$

Comparison of (26) and (28) then reveals that

$$\sum_l a_{ln} a_{ln'}^* = r_n \, \delta_{nn'} \tag{29}$$

If we next define normalization constant a_n and unit vector φ_n^{12} such that

$$a_n \varphi_n^{12} = \sum_m a_{mn} \alpha_m^{12} \tag{30}$$

the arbitrary ovum state takes the *correlated* form

$$\Psi^{123} = \sum_n a_n \varphi_n^{12} \varphi_n^3 \tag{31}$$

Using (29) to help find the normalization constant a_n in (30), we obtain the relation

$$a_n = r_n e^{if_n} \tag{32}$$

where f_n is an arbitrary phase. It follows also from (29) and (30) that the unit vectors $\{\varphi_n^{12}\}$ are mutually orthogonal. Since $r_n > 0$ for all n, (32) implies that $a_n \neq 0$ for all n and therefore *every term* in the expansion (31) is nonzero. The orthonormal set $\{\varphi_n^{12}\}$ stands in one-to-one correspondence with the $\mathscr{H}^3$ basis $\{\varphi_n^3\}$; thus the $\{\varphi_n^{12}\}$ as defined in (30) constitutes a basis for $\mathscr{H}^{12}$. Accordingly, any vector in $\mathscr{H}^{12}$ may be expanded in terms of $\{\varphi_n^{12}\}$.

In Section 2 we used a procedure due to Schrödinger for expressing an arbitrary density operator as a mixture including any desired pure constituent chosen (almost) arbitrarily. A similar procedure, also described in general by Schrödinger, can be applied in our present situation to express the arbitrary ovum state (23) and (31) in a special correlated form that involves any arbitrarily chosen state of $\mathscr{H}^{12}$. The following steps are involved:

(1) Choose any desired state of $\mathscr{H}^{12}$; for the points we wish to make later, we choose the desideratum state defined in (25). Like any vector in $\mathscr{H}^{12}$, it is expressible in terms of $\{\varphi_n^{12}\}$:

$$\hat{\Psi}^{12} = \sum_n c_n \varphi_n^{12} \tag{33}$$

(2) Define the unit vector

$$\gamma_1^{12} = \sum_n g_{1n} \varphi_n^{12} \tag{34}$$

where

$$g_{1n} = \left(\sum_l \frac{|c_l|^2}{r_l} \right)^{-1/2} \frac{c_n}{\sqrt{r_n}}$$

Note that γ_1^{12} is well defined since $r_n > 0$, for all n, a consequence of the restriction that Ψ^{123} be an ovum state. Since all vectors are normalized to unity, we have also the relation

$$c_n = \left(\sum_l r_l |g_{1l}|^2 \right)^{-1/2} \sqrt{r_n}\, g_{1n} \tag{35}$$

(3) Beginning with γ_1^{12}, construct a complete orthonormal set of vectors spanning $\mathscr{H}^{12}$: $\{\gamma_1^{12}, \ldots, \gamma_j^{12}, \ldots\}$.

(4) Define the unit vectors

$$\eta_j^3 = \sum_n e^{if_n} \langle \gamma_j^{12} | \varphi_n^{12} \rangle \, \varphi_n^3 \tag{36}$$

Since $\{\gamma_j^{12}\}$ and $\{\varphi_n^{12}\}$ are both complete orthonormal sets in $\mathscr{H}^{12}$, it is easy to verify that $\{\eta_j^3\}$, like $\{\varphi_n^3\}$, constitutes a new orthonormal basis for $\mathscr{H}^3$. Solving (36) for φ_n^3 and substituting into (31), we obtain

$$\Psi^{123} = \sum_j \sqrt{w_j}\, \Phi_j^{12} \eta_j^3 \tag{37}$$

where

$$w_j = \sum_l r_l |\langle \varphi_l^{12} | \gamma_j^{12} \rangle|^2$$

and

$$\Phi_j^{12} = \sum_n \sqrt{\frac{r_n}{w_j}} \langle \varphi_n^{12} | \gamma_j^{12} \rangle \, \varphi_n^{12}$$

Comparison with (33), (34), and (35) shows that

$$w_1 = \sum_l r_l |g_{1l}|^2 \tag{38}$$

and

$$\Phi_1^{12} = \sum_l c_n \varphi_n^{12} = \hat{\Psi}^{12} \tag{39}$$

(5) The arbitrary ovum state is now expressed in a correlated form involving the desideratum state:

$$\Psi^{123} = \sqrt{w_1} \, \hat{\Psi}^{12} \eta_1^3 + \sum_{j \neq 1} \sqrt{w_j} \, \Phi_j^{12} \eta_j^3 \tag{40}$$

Inspection of (38) shows that $w_1 = 0$ would be possible only if all $g_{1l} = 0$ (since all $r_l > 0$). But according to (34) this would make $c_n = 0$, which in turn would imply via (33) that $\hat{\Psi}^{12} = 0$. This contradicts the original definition (25) of the desideratum state; hence $w_1 \neq 0$.

Unlike correlated forms most commonly encountered in EPR discussions (e.g., the ubiquitous singlet state), the expression (40) involves one orthogonal set $\{\eta_j^3\}$ and one usually nonorthogonal set $\{\Phi_j^{12}\}$, which is therefore not interpretable as the eigenvector set for any observable of $\mathscr{H}^{12}$. However, from (40) we can still compute certain interesting joint probability distributions. Among the observables associated with $\mathscr{H}^3$ and hence measurable by X^3 is one represented by the projector $|\eta_1^3\rangle\langle\eta_1^3|$; similarly, a legitimate observable of $\mathscr{H}^{12}$ is the projector $|\hat{\Psi}^{12}\rangle\langle\hat{\Psi}^{12}|$. Both of these observables are quantal propositions, whose eigenvalues (1, 0) may be interpreted as replies ("yes," "no") to the observable questions they represent.

Because the propositions $|\hat{\Psi}^{12}\rangle\langle\hat{\Psi}^{12}|$ and $|\eta_1^3\rangle\langle\eta_1^3|$ are operators on $\mathscr{H}^{12}$ and $\mathscr{H}^3$ respectively, they obviously commute. We may therefore identify the mean value of their product as the joint probability $W(1, 1)$ for obtaining "yes" answers when the two questions are asked simultaneously; hence in the pure state (40),

$$\begin{aligned} W(1, 1) &= \langle \Psi^{123} | (|\hat{\Psi}^{12}\rangle \langle \hat{\Psi}^{12}|)(|\eta_1^3\rangle \langle \eta_1^3|) | \, \Psi^{123} \rangle \\ &= |\langle \hat{\Psi}^{12} \eta_1^3 | \, \Psi^{123} \rangle|^2 = w_1 \end{aligned} \tag{41}$$

Note that the last step follows entirely from the orthogonality of the $\{\eta_j^3\}$, whether or not the $\{\Phi_j^{12}\}$ are orthogonal. Similarly, we may easily compute the joint probability $W(0, 1)$ that question $|\hat{\Psi}^{12}\rangle\langle\hat{\Psi}^{12}|$ will yield "no" while $|\eta_1^3\rangle\langle\eta_1^3|$ answers "yes." In quantum logic, the negation of $|\hat{\Psi}^{12}\rangle\langle\hat{\Psi}^{12}|$ is $(\mathbf{1}^{12} - |\hat{\Psi}^{12}\rangle\langle\hat{\Psi}^{12}|)$, so that in pure state (40)

$$\begin{aligned} W(0, 1) &= \langle\Psi^{123}|(\mathbf{1}^{12} - |\hat{\Psi}^{12}\rangle\langle\hat{\Psi}^{12}|)(|\eta_1^3\rangle\langle\eta_1^3|)|\,\Psi^{123}\rangle \\ &= w_1 - w_1 = 0 \end{aligned} \tag{42}$$

where again we note that the result depends only on the orthogonality of $\{\eta_j^3\}$.

Using as raw material the arbitrary ovum state of $\mathscr{H}^{123}$, X^3 can exploit the probability relations (41) and (42) to generate an $\mathscr{H}^{12}$ ensemble of systems in the desideratum state $\hat{\Psi}^{12}$ without interacting with any $\mathscr{H}^{12}$. The procedure is simple: X^3 measures on $\mathscr{H}^3$ the proposition $|\eta_1^3\rangle\langle\eta_1^3|$; if the result is "yes," X^3 accepts the corresponding $\mathscr{H}^{12}$ into the ensemble being generated; otherwise, $\mathscr{H}^{12}$ is rejected.

To demonstrate rigorously (i.e., without recourse to such fantasy as wave packet reduction) that this repeatable procedure would actually produce an ensemble of $\mathscr{H}^{12}$ in the state $\hat{\Psi}^{12}$, we note first that, since $w_1 \neq 0$, X^3 will definitely obtain "yes" answers on occasion; i.e., the procedure will generate an $\mathscr{H}^{12}$ ensemble. From (42) it follows that each $\mathscr{H}^{12}$ accepted by the procedure is in a condition such that a measurement of $|\hat{\Psi}^{12}\rangle\langle\hat{\Psi}^{12}|$ cannot yield a "no," i.e., must yield "yes." Thus an observer of $\mathscr{H}^{12}$ who happened to ask the question $|\hat{\Psi}^{12}\rangle\langle\hat{\Psi}^{12}|$ while attempting to determine the quantum state ρ^{12} would necessarily obtain the condition

$$1 = \mathrm{Tr}_{12}(\rho^{12}|\hat{\Psi}^{12}\rangle\langle\hat{\Psi}^{12}|) = \langle\hat{\Psi}^{12}|\rho^{12}|\hat{\Psi}^{12}\rangle \tag{43}$$

The only positive operator satisfying (43) is

$$\rho^{12} = |\hat{\Psi}^{12}\rangle\langle\hat{\Psi}^{12}| \tag{44}$$

Naturally an observer not so fortunate as to discover property (43) right away would eventually, after gathering much quorum data, also arrive at (44).

Let us return now to observer X^1, who studies only $\mathscr{H}^1$. If (44) is the state of $\mathscr{H}^{12}$, then the reduced density operator ρ^1 for $\mathscr{H}^1$, the state that X^1 would find from analysis of his quorum measurements, is easily obtained from (25) and (44):

$$\rho^1 = \mathrm{Tr}_2\,\rho^{12} = \sum_j \hat{p}_j|\psi_j^1\rangle\langle\psi_j^1| = \hat{\rho}^1(\beta^1) \tag{45}$$

the same canonical state (3) prepared in earlier scenarios (a) by mixing and (b) by equilibration.

We describe the present method as (c) subensemble *selection*, since X^3 has selected out of all $\mathscr{H}^{123}$ systems in the arbitrary ovum state a subset whose $\mathscr{H}^1$ components (undisturbed by X^3) will be found by X^1 to constitute the canonical ensemble characterized by (3).

The thermodynamic implications of this selection method of generating a canonical ensemble are as bizarre as in the mixing case. If X^1 applies normal thermodynamic reasoning, he will conclude from (17) that the adiabatic availability of the system $\mathscr{H}^1$ is zero; i.e., each system $\mathscr{H}^1$ in the ensemble can be harnessed to do, on average, no work on another system $\mathscr{H}^0$.

It can be seen from the structure of (25) that there are correlations inherent in desideratum state $\hat{\Psi}^{12}$ that would permit X^2 to make additional subensemble selections from the present canonical ensemble of systems $\mathscr{H}^1$. To be specific, it follows from (25) that the joint probability that measurements of $|\psi_j^1\rangle\langle\psi_j^1|$ by X^1 and $|\psi_j^2\rangle\langle\psi_j^2|$ by X^2 will both yield "yes" has nonzero value $\hat{p}_j$, and the probability that X^1 would obtain "no" while X^2 obtains "yes" is zero. X^2 has therefore the power to filter unobtrusively the canonical ensemble being provided to X^1. The method is patterned after that used by X^3 to generate the desideratum state: X^2 measures on $\mathscr{H}^2$ the proposition $|\psi_j^2\rangle\langle\psi_j^2|$; if the result is "yes," X^2 accepts the corresponding $\mathscr{H}^1$; otherwise, $\mathscr{H}^1$ is rejected. An argument parallel to that leading to (44) would show that the ensemble of systems $\mathscr{H}^1$ generated by this method would be found by X^1 to be described by the pure quantum state ψ_j^1, which was originally defined in (6) as an eigenstate of H^1, the energy of system $\mathscr{H}^1$.

Consider the thermodynamic significance of this result. The systems $\mathscr{H}^1$ now being studied by X^1 comprise a subset of the very same systems that participated in the earlier canonical ensemble generated for X^1 by X^3; X^2, like X^3, engaged in no physical interaction with $\mathscr{H}^1$. But, applying the thermodynamic principle (18), X^1 would now conclude that the systems $\mathscr{H}^1$ with which he now deals have adiabatic availability

$$\mathscr{A}(|\psi_j^1\rangle\langle\psi_j^1|) = E_j - E_0 \tag{46}$$

which is strictly positive unless ψ_j^1 is the ground state, an exception easily avoided by selector X^2 in his free choice of j. Consequently, thermodynamic reasoning would here lead X^1 to believe that each system $\mathscr{H}^1$ in the present ensemble can be harnessed to perform, on average, *positive* work on another system $\mathscr{H}^0$.

From a slightly different perspective, we may say that X^2, without disturbing the systems $\mathscr{H}^1$ in the canonical ensemble, can *partition* that ensemble into pure subensembles with states $\{\psi_j^1\}$, *each* of which contains systems $\mathscr{H}^1$ that must be regarded thermodynamically as being capable of doing, on average, nonnegative work on auxiliary systems $\mathscr{H}^0$. We have therefore a complete paradox: the same ensemble of systems $\mathscr{H}^1$ that are capable of doing zero average work can be split into subensembles each comprised of systems capable of doing nonzero average work.

It is as though the output F of a refinery consisted of N barrels of fuel with the following property: when used to power N cranes, F lifts N weights through an average displacement of *zero*; but when the *same* N barrels are partitioned into subsets $\{F_j\}$, where F_j contains N_j barrels, each F_j lifts N_j weights through a *positive* average displacement, with the contradictory complication that the F indeed lifted all N weights through a *nonzero* average displacement.

Surely thermodynamic availability for $\mathscr{H}^1$ cannot depend so capriciously upon the whims of disconnected observers like X^2 and X^3. Yet this entire selection scheme has been constructed from standard principles of quantum mechanics and normal thermodynamic reasoning based upon formulas commonly used in quantum statistics to represent such thermodynamic notions as entropy and equilibrium states. It would seem, therefore, that some aspect of quantum mechanics or thermodynamics or both requires modification.

5. THERMODYNAMIC CAUSALITY

Whether a canonical ensemble of systems $\mathscr{H}^1$ is prepared by (a) mixing, (b) equilibration, or (c) selection is not significant in quantum mechanics; for in all cases the density operator (3) is the same. Thus no data X^1 can accumulate by measuring quantal observables of $\mathscr{H}^1$ can distinguish the three cases.

With respect to (c) selection, a process involving correlations between $\mathscr{H}^1$ and $\mathscr{H}^2$, and between $\mathscr{H}^{12}$ and $\mathscr{H}^3$, it might be argued that the canonical state determined by X^1 is "improper"(5) since only the whole composite $\mathscr{H}^{123}$ should be regarded as having a state. After all, X^1's attribution of state (3) to $\mathscr{H}^1$ alone ignores real correlations actually exploited by X^2 and X^3 in their roles as preparer-demons. These correlations were not discoverable by X^1 because of his limited repertoire of measurement devices. By contrast, if thermodynamics is regarded as fundamental, then in case (b) equilibration, there is nothing quantum-mechanical about $\mathscr{H}^1$ that is hidden from X^1. The canonical density operator (3) would be indeed the true, unambiguous state of $\mathscr{H}^1$.

Thus it seems that our quantal observer X^1 of system $\mathscr{H}^1$ who through quorum measurements of $\mathscr{H}^1$ observables has determined the state (3) has two ways—both somewhat unpalatable—to attempt to distinguish (b) and (c). He can try to determine whether the systems $\mathscr{H}^1$ can do work on auxiliary systems $\mathscr{H}^0$ or he can try to ascertain whether $\mathscr{H}^1$ is correlated with any other system in the universe! As we have already seen, the thermodynamic availability approach would suffice to distinguish methods (a) and (b).

Another possibility for escaping from these dilemmas begins with the recognition that they are but part of a more general theoretical tension that exists between quantum mechanics and thermodynamics. The standard quantal time evolution of an isolated system is described by a unitary transformation that leaves invariant not only the mean energy (13) but also the entropy (14). Such motion, a universal trait of quantum mechanics, is incompatible with commonplace irreversible processes in which entropy increases. If, however, the quantal law of motion were generalized to encompass energy-conserving, entropy-increasing state evolution, the generalized law would provide a foundation for a consistent quantum thermodynamics. A particularly comprehensive effort to construct such a theory is embodied in the Beretta equation.(6,7)

The details of quantum thermodynamics are presently unknown, and the very idea of taking thermodynamics to be so fundamental is controversial. Nevertheless, it is interesting to explore the consequences, for the problems raised in the present study, of a hypothetical law of motion featuring short-term unitary evolution but longer-term thermodynamic equilibration (entropy maximization subject to constraints). We proceed by assuming, as in Section 4, that $\mathscr{H}^1$, $\mathscr{H}^2$, and $\mathscr{H}^3$ are not interacting. However, since in the selection scheme X^3 performs a measurement on $\mathscr{H}^3$, we shall introduce an apparatus system $\mathscr{H}^4$ with which $\mathscr{H}^3$ will interact. The energy operators for the whole system and its various subsystems are therefore related as follows:

$$H^{1234}=H^{12}+H^{34} \tag{47}$$

where $H^{12}=H^1+H^2$, but H^{34} is not an additive form due to the interaction between $\mathscr{H}^3$ and $\mathscr{H}^4$. If $\mathscr{H}^4$ is initially in the state χ_0^4 and, as before, $\mathscr{H}^{123}$ is in the arbitrary ovum state Ψ^{123}, then ordinary unitary evolution of $\mathscr{H}^{1234}$ occurring as $\mathscr{H}^3$ interacts with apparatus $\mathscr{H}^4$ leads to the post-measurement-interaction state given, using the notation of (37), by

$$e^{-it(H^{12}+H^{34})}\,\Psi^{123}\chi_0^4=\sum_j\sqrt{w_j}\,\Phi_j^{12}(t)\,\Psi_j^{34}(t) \tag{48}$$

where

$$\Phi_j^{12}(t)=e^{-itH^{12}}\,\Phi_j^{12}$$

and

$$\Psi_j^{34}(t) = e^{-itH^{34}}(\eta_j^3 \chi_0^4)$$

Since $\{\eta_j^3\}$ is an orthogonal set, the reduced density operator for $\mathscr{H}^{12}$ at time t takes the form

$$\begin{aligned} \rho^{12}(t) &= \sum_j w_j |\Phi_j^{12}(t)\rangle\langle\Phi_j^{12}(t)| \\ &= e^{-itH^{12}}\rho^{12}(0)\,e^{itH^{12}} \end{aligned} \tag{49}$$

a form clearly illustrating that, absent interaction between $\mathscr{H}^{12}$ and $\mathscr{H}^{34}$, $\mathscr{H}^{12}$ evolves as though $\mathscr{H}^{34}$ did not exist.

As described earlier, X^3 uses measurement results (obtained in the $\mathscr{H}^{34}$ interaction process) to select out, via correlation, the $j=1$ subensemble of the $\mathscr{H}^{12}$ ensemble described now by (49). That subensemble is characterized by the desideratum state (39) which is now seen from (49) to be evolving unitarily according to

$$\hat{\Psi}^{12}(t) = e^{-it(H^1+H^2)}\,\hat{\Psi}^{12} \tag{50}$$

After substituting (25) into (50), we obtain as the reduced density operator for $\mathscr{H}^1$ at time t

$$\rho^1(t) = \frac{e^{-\beta^1 H^1}}{Z(\beta^1)} = \hat{\rho}^1(\beta^1) \tag{51}$$

the familiar canonical state, which is time-independent.

Admittedly, none of this is especially surprising; as expected, it turns out in (normal) quantum mechanics that the actions of X^3 in performing the needed measurements are not detectable by X^1, who determines from his quorum measurements the stationary canonical state (51), the same state description he would have found had $\mathscr{H}^1$ been prepared by equilibration.

Suppose, however, that the conventional unitary time evolution were only an excellent short-term approximation, that the true causal evolution for any isolated system were instead an energy-conserving but entropy-maximizing (hence nonunitary) transformation. In such a world, the system $\mathscr{H}^{1234}$ would eventually equilibrate to a final state in the form of a tensor product of canonical density operators:

$$\rho^{1234}(\infty) = \hat{\rho}^1(\bar{\beta}^1)\,\hat{\rho}^2(\bar{\beta}^2)\,\hat{\rho}^{34}(\bar{\beta}^{34}) \tag{52}$$

where $\bar{\beta}^j$ is determined by

$$\mathrm{Tr}_j(\hat{\rho}^j(\bar{\beta}^j)\,H^j) = \langle\Psi^{123}\chi_0^4|H^j|\,\Psi^{123}\chi_0^4\rangle \tag{53}$$

In other words, each isolated system $\mathscr{H}^1$, $\mathscr{H}^2$, and $\mathscr{H}^{34}$ assumes the state of maximum entropy consistent with its total (conserved) initial energy. From (52) we obtain immediately the reduced density operator for $\mathscr{H}^{12}$

$$\rho^{12}(\infty) = \hat{\rho}^1(\bar{\beta}^1)\,\hat{\rho}^2(\bar{\beta}^2) \tag{54}$$

which is obviously very different from the usual result (49). Moreover, (52) embodies *no correlations* between $\mathscr{H}^{12}$ and $\mathscr{H}^{34}$, so that even if X^3, using apparatus $\mathscr{H}^4$, makes his measurements and associated selections, the $\mathscr{H}^{12}$ subensemble thus selected will be nothing but a random sample from the whole $\mathscr{H}^{12}$ ensemble. Presumably X^1 would assign to such a subensemble the reduced density operator for $\mathscr{H}^1$ implied by (54):

$$\rho^1(\infty) = \hat{\rho}^1(\bar{\beta}^1) \tag{55}$$

In stark contrast to (51), where β^1 was *arbitrarily* chosen when the desideratum state was constructed, in (55) $\bar{\beta}^1$ was determined via (53) by the mean energy of $\mathscr{H}^1$ in the initial state of $\mathscr{H}^{1234}$. Such a comparison serves to underscore the peculiarities of quantal probability relations that Schrödinger found repugnant. Given an arbitrary ovum state, X^3 arbitrarily chooses to measure some observable that will enable him to prepare by selection any desideratum state of his choice. Never interacting with $\mathscr{H}^1$, X^3 predetermines, by choosing a value for β^1, the mean energy that X^1 will assign to the selected canonical ensemble of systems $\mathscr{H}^1$. Though formally canonical, surely such an ensemble cannot reasonably represent the concept of thermodynamic equilibrium; but how can X^1 know this?

If, as we are here conjecturing, the unitary evolution is a relatively short-term phenomenon, then the state (51) would not actually be stationary but would evolve eventually to (55), a process that would be detectable by X^1 exclusively through measurements within his repertoire. Through such studies X^1 could in fact distinguish a canonical ensemble of systems $\mathscr{H}^1$ prepared in the (unambiguous) manner (b) equilibration from ensembles prepared by the (ambiguous) method (c) selection. Similarly, X^1 should probably be able to differentiate canonical ensembles generated by (a) mixing and (b) equilibration by waiting; the ensemble in case (a) would eventually relax to a collection of systems with many different temperatures, whereas the unambiguous ensemble in case (b) would be stationary.

We therefore conclude that a generalized quantal law of motion, designed for compatibility with fundamental thermodynamic principles, would provide also a means for resolving paradoxes associated with the characteristic ambiguity of ensembles in quantum mechanics.

REFERENCES

1. E. Schrödinger, *Proc. Cambridge Philos. Soc.* **31**, 555 (1935).
2. E. Schrödinger, *Proc. Cambridge Philos. Soc.* **32**, 446 (1936).
3. J. L. Park and W. Band, *Found. Phys.* **1**, 211 (1971).
4. G. N. Hatsopoulos and E. P. Gyftopoulos, *Found. Phys.* **6**, 15, 127, 439, 561 (1976).
5. B. d'Espagnat, *Conceptual Foundations of Quantum Mechanics* (Benjamin, London, 1976), p. 58.
6. G. P. Beretta, E. P. Gyftopoulos, J. L. Park, and G. N. Hatsopoulos, *Nuovo Cimento B* **82**, 169 (1984).
7. G. P. Beretta, E. P. Gyftopoulos, and J. L. Park, *Nuovo Cimento B* **87**, 77 (1985).

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